Orbit-averaged guiding-center Fokker-Planck operator for numerical applications in axisymmetric toroidal plasmas

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The gyromotion of particles in toroidal plasmas is characterized by a Larmor radius ρ generally much smaller than the scale-length L_B of the magnetic field non-uniformity. In this case the particle orbits can be parametrized by a constant of the motion - the energy $\mathcal{E}$ - and one adiabatic invariant, the magnetic moment μ. Using the ordering $\epsilon_B \equiv \rho/L_B \ll 1$, Lie-transforms methods were applied to derive a guiding-center (GC) Fokker-Planck (FP) operator in the local GC coordinates $(\mathbf{X}, \mathcal{E}, \mu, \varphi)$, where $\mathbf{X}$ is the GC position and φ is the gyroangle [1]. This transformation was applied to the case of a collision operator with an isotropic and uniform field particles distribution [1].

For axisymmetric plasmas where the toroidal canonical momentum P_ϕ is also an invariant of the motion, an equivalent CG FP operator was derived in a new set of GC coordinates $Z^\alpha = (\bar{\psi}, \theta, p, \xi_0)$ that are well-suited for numerical applications [2]. Here the flux-surface label $\bar{\psi} \equiv -cP_\phi/e$ and the particle momentum $p \equiv \sqrt{2m\mathcal{E}}$ are constant of the motion, the pitch-angle coordinate ξ_0 defined as

$$\xi_0(\bar{\psi}, \mathcal{E}, \mu) \equiv \left\{ \begin{array}{ll} \sqrt{1 - \mu B_0(\bar{\psi})/\mathcal{E}}. & \text{for trapped particle orbits} \\ \sigma\sqrt{1 - \mu B_0(\bar{\psi})/\mathcal{E}}. & \text{for passing particle orbits} \end{array} \right| \tag{1}$$

is an adiabatic invariant, and the poloidal angle θ parametrizes the position along the GC orbit. In (1), $B_0(\psi)$ is the minimum value of the magnetic field amplitude on the flux-surface ψ.

In the present paper, the GC FP operator is transformed such that it can commute with the orbit-averaging operation. In the low-collisionality regime, a 3-D bounce-averaged FP equation is thus obtained in the space of invariants $I^a = (\bar{\psi}, p, \xi_0)$. In addition, a FP collision operator with a non-uniform field particles distribution is explicitly derived in the I^a coordinates. Simplified expression are also obtained in the thin-orbit width approximation characterized by $\epsilon_\psi \equiv \epsilon_B q/\varepsilon \ll 1$, where q is the safety factor and $\varepsilon = r/R$ is the local inverse aspect ratio.

GC FP equation in Z^α coordinates

The GC FP evolution equation in Z^α coordinates is given by [2]

$$\epsilon_\tau \frac{\partial F}{\partial \tau} + \dot{\theta}\frac{\partial F}{\partial \theta} = \epsilon_\nu C_{\mathrm{gc}}(F) \tag{2}$$

where $F = F(\tau, \bar{\psi}, \theta, p, \xi_0)$ is the GC distribution function, $\dot{\theta}$ characterizes the poloidal GC motion, $\epsilon_\nu = L_B/\lambda_\nu$ where λ_ν is the mean-free pass, and the collision operator is

$$C_{\rm gc}(F) = -\frac{1}{\mathcal{J}}\frac{\partial}{\partial Z^\alpha}\left[\mathcal{J}\left(K^\alpha_{\rm gc}F - D^{\alpha\beta}_{\rm gc}\frac{\partial F}{\partial Z^\beta}\right)\right] \tag{3}$$

where $\mathcal{J}$ is the Jacobian of the transformation, and $K^\alpha_{\rm gc}$ and $D^{\alpha\beta}_{\rm gc}$ are the GC convection and diffusion coefficients, respectively. They are obtained from particle convection and diffusion coefficients using the transformation $\mathbf{T}^{-1}$ from particle to GC space and the projection vectors $\boldsymbol{\Delta}^\alpha$ to Z^α coordinates according to [1]

$$\begin{aligned} K^\alpha_{\rm gc} &= \left\langle \mathbf{T}^{-1}\mathbf{K}\cdot\boldsymbol{\Delta}^\alpha\right\rangle_g \\ D^{\alpha\beta}_{\rm gc} &= \left\langle \left(\boldsymbol{\Delta}^\alpha\right)^{\rm T}\cdot\mathbf{T}^{-1}\mathbb{D}\cdot\boldsymbol{\Delta}^\beta\right\rangle_g \end{aligned} \tag{4}$$

where $\langle\cdots\rangle_g$ denotes gyro-averaging.

The GC projection vectors in $I^a = (\bar{\psi}, p, \xi_0)$ space are given by

$$\begin{aligned} \boldsymbol{\Delta}^{\bar{\psi}} &= \epsilon_B\frac{\hat{\mathbf{b}}}{m\Omega}\times\boldsymbol{\nabla}\bar{\psi} - \epsilon_\psi\frac{\delta\psi}{p^2\xi^2}\mathbf{p}_\epsilon + \epsilon_\psi\frac{\Omega\delta\psi}{2\mu B}\frac{1-\xi^2}{\xi^2}\frac{\partial\boldsymbol{\rho}_\epsilon}{\partial\varphi} + \mathcal{O}(\epsilon_B^2) \\ \boldsymbol{\Delta}^p &= \frac{\mathbf{p}_\epsilon}{p} + \mathcal{O}(\epsilon_B^2) \\ \boldsymbol{\Delta}^{\xi_0} &= \frac{1-\xi_0^2}{2\xi_0}\left(\frac{\mathbf{p}_\epsilon}{m\mathcal{E}} - \frac{\Omega}{\mu B}\frac{\partial\boldsymbol{\rho}_\epsilon}{\partial\varphi} - \boldsymbol{\Delta}^{\bar{\psi}}\frac{d\ln B_0}{d\bar{\psi}}\right) + \mathcal{O}(\epsilon_B^2) \end{aligned} \tag{5}$$

where $\mathbf{p}_\epsilon = \mathbf{T}^{-1}\mathbf{p}$, $\boldsymbol{\rho}_\epsilon = \mathbf{T}^{-1}\boldsymbol{\rho}$, Ω is the gyrofrequency, $\xi = \sigma\sqrt{1-(1-\xi_0^2)B/B_0(\bar{\psi})}$ is the local pitch-angle coordinate, and $\delta\psi = \psi - \bar{\psi}$ is the local orbit width.

Orbit-averaged GC FP equation

The orbit-averaging (OA) operation is expressed as

$$\langle\cdots\rangle_O = \frac{1}{\tau_O}\oint_O\frac{d\theta}{\dot{\theta}}\cdots \tag{6}$$

where τ_O is the orbit time. We formally introduce the function

$$\mathcal{G}(\theta, \bar{\psi}, p, \xi_0) \equiv \frac{\sigma\mathcal{J}\dot{\theta}\tau_O}{2\pi\mathcal{J}_O} \tag{7}$$

where the OA-Jacobian $\mathcal{J}_O \equiv \tau_O v p^2|\xi_0|/(2\pi B_0(\bar{\psi}))$ and τ_O are invariants of the GC motion.

We find that $\mathcal{G}(\theta, \bar{\psi}, p, \xi_0) = 1 + \mathcal{O}(\epsilon_\psi\epsilon_B)$, meaning that the θ-dependence in the product $\mathcal{J}\dot{\theta}$ is of order $\epsilon_\psi\epsilon_B$. In the general case where $\epsilon_\psi \sim 1$, these θ-dependent corrections in $\mathcal{G}$ are of the same order as the GC corrections in $K^\alpha_{\rm gc}$ and $D^{\alpha\beta}_{\rm gc}$. Yet, it is possible to transform the GC FP equation (2) such that it commutes with the orbit-averaging operation (6), which yields, in the low-collisionality regime $\epsilon_\nu \ll 1$

$$\epsilon_\tau \frac{\partial F^{(0)}}{\partial \tau} = \left\langle C_{\text{gc}}\left(F^{(0)}\right)\right\rangle_O \equiv -\epsilon_\nu \frac{1}{\mathcal{J}_O}\frac{\partial}{\partial I^a}\left(\mathcal{J}_O\left[K_{\text{gc}}^{a(0)}F^{(0)} - D_{\text{gc}}^{ab(0)}\frac{\partial F^{(0)}}{\partial I^b}\right]\right) + \epsilon_\nu M_{\text{gc}}^{(0)}F^{(0)} \tag{8}$$

with

$$\begin{aligned} K_{\text{gc}}^{a(0)} &= \left\langle K_{\text{gc}}^{a}\right\rangle_O + \left\langle D_{\text{gc}}^{a\beta}\mathcal{G}\frac{\partial \mathcal{G}^{-1}}{\partial Z^\beta}\right\rangle_O \\ D_{\text{gc}}^{ab(0)} &= \left\langle D_{\text{gc}}^{ab}\right\rangle_O \\ M_{\text{gc}}^{(0)} &= \frac{1}{\mathcal{J}_\mathcal{O}}\frac{\partial}{\partial I^a}\left[\mathcal{J}_O\left\langle D_{\text{gc}}^{a\beta}\mathcal{G}\frac{\partial \mathcal{G}^{-1}}{\partial Z^\beta}\right\rangle_O\right] + \left\langle K_{\text{gc}}^{\alpha}\mathcal{G}\frac{\partial \mathcal{G}^{-1}}{\partial Z^\alpha}\right\rangle_O \end{aligned} \tag{9}$$

where $F = F^{(0)}(\tau, \bar{\psi}, p, \xi_0)$ is independent of θ. According to the equation (8) the distribution thus evolves on the collision time scale : $\epsilon_\tau = \epsilon_\nu$.

In the thin-orbit width approximation where $\epsilon_\psi \ll 1$, (8) simplifies as $M_{\text{gc}}^{(0)} = 0$ and $K_{\text{gc}}^{a(0)} = \left\langle K_{\text{gc}}^{a}\right\rangle_O$.

GC collision operator with isotropic non-uniform field particles

The particle convection and diffusion coefficients for a collision operator with isotropic field particles can be expressed as [1]

$$\begin{aligned} \mathbf{K} &= -\nu \mathbf{p} \\ \mathbb{D} &= D_t\left(\mathbb{I} - \frac{\mathbf{pp}}{p^2}\right) + D_l\frac{\mathbf{pp}}{p^2} \end{aligned} \tag{10}$$

where the convection coefficient ν and diffusion coefficients $D_\parallel$ and $D_\perp$ are function of $(p, \mathbf{x})$ only such that the transformation to GC coefficients yields [1]

$$\begin{aligned} \nu_\epsilon(\mathbf{Z}) &= \nu(p, \mathbf{X} + \boldsymbol{\rho}_\epsilon) = \nu(p, \mathbf{X}) + \epsilon_{n,T}\boldsymbol{\rho}_\epsilon \cdot \boldsymbol{\nabla}\nu \\ D_{t\epsilon}(\mathbf{Z}) &= D_t(p, \mathbf{X} + \boldsymbol{\rho}_\epsilon) = D_t(p, \mathbf{X}) + \epsilon_{n,T}\boldsymbol{\rho}_\epsilon \cdot \boldsymbol{\nabla}D_t \\ D_{l\epsilon}(\mathbf{Z}) &= D_l(p, \mathbf{X} + \boldsymbol{\rho}_\epsilon) = D_l(p, \mathbf{X}) + \epsilon_{n,T}\boldsymbol{\rho}_\epsilon \cdot \boldsymbol{\nabla}D_l \end{aligned} \tag{11}$$

since the particle momentum and GC momentum are identical up to order ϵ_B. We can assume from now on that $\epsilon_B \sim \epsilon_{n,T} \sim \epsilon$.

Orbit-averaged GC collision operator for numerical applications

In the thin-orbit approximation $\epsilon_\psi \ll 1$, the orbit-averaged GC FP operator (8) can be expressed under the following conservative form [2], which is adequate for numerical implementation in a 3-D FP code

$$\left\langle C_{\text{gc}}\left[f^{(0)}\right]\right\rangle_O = -\frac{1}{\mathcal{J}_O}\frac{\partial}{\partial \bar{\psi}}\left(\mathcal{J}_O\left\|\nabla\bar{\psi}\right\|_0 S_{\text{L}}^{\bar{\psi}}\right) - \frac{1}{\mathcal{J}_O}\frac{\partial}{\partial p}\left(\mathcal{J}_O S_{\text{L}}^{p}\right) + \frac{1}{\mathcal{J}_O p}\frac{\partial}{\partial \xi_0}\left(\mathcal{J}_O\sqrt{1-\xi_0^2}S_{\text{L}}^{\xi_0}\right) \tag{12}$$

The I^a space fluxes are expressed in terms of OA-GC convection and diffusion coefficients

$$\begin{pmatrix} S_{\mathrm{L}}^{\bar\psi} \\ S_{\mathrm{L}}^{p} \\ S_{\mathrm{L}}^{\xi_0} \end{pmatrix} \equiv \begin{pmatrix} K_{\mathrm{L}}^{\bar\psi} \\ K_{\mathrm{L}}^{p} \\ K_{\mathrm{L}}^{\xi_0} \end{pmatrix} F^{(0)} - \begin{pmatrix} D_{\mathrm{L}}^{\bar\psi\bar\psi} & D_{\mathrm{L}}^{\bar\psi p} & D_{\mathrm{L}}^{\bar\psi\xi_0} \\ D_{\mathrm{L}}^{p\bar\psi} & D_{\mathrm{L}}^{pp} & D_{\mathrm{L}}^{p\xi_0} \\ D_{\mathrm{L}}^{\xi_0\bar\psi} & D_{\mathrm{L}}^{\xi_0 p} & D_{\mathrm{L}}^{\xi_0\xi_0} \end{pmatrix} \begin{pmatrix} \left\|\nabla\bar\psi\right\|_0 \partial/\partial\bar\psi \\ \partial/\partial p \\ -p^{-1}\sqrt{1-\xi_0^2}\partial/\partial\xi_0 \end{pmatrix} F^{(0)} \tag{13}$$

For collisions with non-uniform isotropic field particle distributions, these coefficients are explicitly derived from (4), (5), (9) and (11)

$$\begin{pmatrix} K_{\mathrm{L}}^{p} \\ D_{\mathrm{L}}^{pp} \end{pmatrix} = \left(1 + \epsilon_\psi \delta\psi^{(0)} \frac{\partial}{\partial\psi}\right) \begin{pmatrix} -\nu p \\ D_l \end{pmatrix} + \mathcal{O}(\epsilon^2, \epsilon_\psi^2, \epsilon\epsilon_\psi) \tag{14}$$

$$\begin{pmatrix} K_{\mathrm{L}}^{\xi_0} \\ D_{\mathrm{L}}^{p\xi_0} \end{pmatrix} = \begin{pmatrix} -\nu p \\ D_l \end{pmatrix} \frac{\sqrt{1-\xi_0^2}}{2\xi_0} \left(\epsilon_\psi \overline{\delta\psi}^{(0)} + \epsilon\lambda_{\mathrm{gc}}^{(0)}\right) + \mathcal{O}(\epsilon^2, \epsilon_\psi^2, \epsilon\epsilon_\psi) \tag{15}$$

$$\begin{pmatrix} K_{\mathrm{L}}^{\bar\psi} \\ D_{\mathrm{L}}^{p\bar\psi} \\ D_{\mathrm{L}}^{\xi_0\bar\psi} \end{pmatrix} = -\epsilon_\psi \frac{\delta\psi^{(0)}}{p\left\|\boldsymbol{\nabla}\bar\psi\right\|_0} \begin{pmatrix} -\nu p \\ D_l \\ -\sqrt{1-\xi_0^2} D_t/\xi_0 \end{pmatrix} + \mathcal{O}(\epsilon^2, \epsilon_\psi^2, \epsilon\epsilon_\psi) \tag{16}$$

$$D_{\mathrm{L}}^{\bar\psi\bar\psi} = \mathcal{O}(\epsilon^2, \epsilon_\psi^2, \epsilon\epsilon_\psi) \tag{17}$$

$$D_{\mathrm{L}}^{\xi_0\xi_0} = \left[\Delta^\dagger - \epsilon\lambda_{\mathrm{gc}}^{(0)\dagger} + \epsilon_\psi \left(\overline{\delta\psi}^{(0)} \frac{(1-\xi_0^2)}{\xi_0^2} + \delta\psi^{(0)\dagger} \frac{\partial}{\partial\psi}\right)\right] D_t + \mathcal{O}(\epsilon^2, \epsilon_\psi^2, \epsilon\epsilon_\psi) \tag{18}$$

where $\delta\psi^{(0)}$, $\delta\psi^{(0)\dagger}$, $\overline{\delta\psi}^{(0)}$, $\lambda_{\mathrm{gc}}^{(0)}$, $\lambda_{\mathrm{gc}}^{(0)\dagger}$ and $\Delta^\dagger$ are bounce-averaging coefficients to be calculated from particle orbits. Note that to leading order we retrieve the usual collision operator in the zero-orbit width limit.

The implementation of this operator in the 3-D Fokker-Planck code LUKE [3] is under way. It will describe neoclassical transport and thus include the bootstrap current consistently with other sources (radio-frequency, ohmic heating) in general current drive calculations.

References

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