Mode-impedance technique for modeling of electromagnetic wave propagation in plasmas

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1. Introduction

In the present communication we propose a relatively simple general technique for modeling the propagation of electromagnetic waves in media with complex dielectric response and spatial dispersion. The basic problem considered below is reconstruction of electromagnetic fields inside and outside a layer of stationary but spatially inhomogeneous plasma highlighted by the known incident radiation. This problem primarily corresponds to the plasma heating by high frequency waves in magnetic traps. Mathematically such processes are naturally described by a boundary problem for Maxwell equations with inhomogeneous constitutive relations. However, there is a regular way to transform this problem to a purely evolutional problem defined only by initial conditions. In this form the problem becomes more convenient both for developing the analytical solutions and for numerical calculations.

The considered approach is a particular form of the more general method of invariant embedding [1]. In most cases the invariant embedding technique has been applied to a scalar wave equation. However, wave propagation in anisotropic and gyrotropic media is more adequately described by vector wave equations. A specific feature of such media is the existence of several normal waves with different polarizations. In this case, a straightforward generalization of the standard invariant embedding technique is possible, but it involves essential technical difficulties. The main goal of the present communication is the demonstration of a new interpretation of the invariant embedding technique. A particular new development concerns the interaction between the normal waves of the studied medium in terms of the evolution of the reflection operator that couples counterpropagating normal waves. This reformulation of the invariant embedding approach gives rise to new evolutionary equations that are more transparent, physically intuitive, and very flexible for further analytical transformations or numerical studies.

2. Mode-impedance technique for Maxwell equations

Let us consider a plane monochromatic wave which is incident to inhomogeneous plasma layer, discussing of generalization of this technique to three-dimensional geometry you can find in [2]. Parameters of the layer vary along the coordinate *z*, and the layer is limited within the range $a \le z \le b$. Outside the layer there is some "external" medium which is assumed to be homogeneous. Therefore the wave field inside and outside the layer may be presented in the following form:

$$\mathbf{E} = \begin{cases} \sum\limits_{\text{all modes}} \mathbf{E}_0 \exp(-i\omega t + ik_x x + ik_z z) + \sum\limits_{\text{all modes}} \hat{R}_0 \mathbf{E}_0 \exp(-i\omega t + ik_x x - ik_z z), & z < a \\ \mathbf{E}(z) \exp(-i\omega t + ik_x x), & a \le z \le b\,. \\ \sum\limits_{\text{all modes}} \hat{T}_0 \mathbf{E}_0 \exp(-i\omega t + ik_x x + ik_z z), & z > b \end{cases} \quad (1)$$

Here $\mathbf{E}_0$, $\hat{R}_0\mathbf{E}_0$ and $\hat{T}_0\mathbf{E}_0$ define, correspondingly, the amplitude and polarization of the incident, reflected and transmitted waves outside the slab, the x axis is directed in the incident plane along the transverse wave vector. In equation (1) we assume for simplicity that dielectric properties of the external medium are identical at the both ends of the slab, thus ω, k_x and k_z satisfy the same dispersion relation. Summation symbol in (1) indicates that in principle several modes with different k_z may exist in the external medium. Note that the polarization vector of reflected and transmitted waves may differ from the those of the incident wave due to excitation of several normal waves in anisotropic and gyrotropic media. This is taken into account by introduction of the reflection and transmission matrixes, $\hat{R}_0$ and $\hat{T}_0$, instead of scalar reflection and transmission coefficients typical of isotropic medium.

Our final goal is to find the wave distribution $\mathbf{E}(z)$ inside the slab and the reflection and transmission matrixes, $\hat{R}_0$ and $\hat{T}_0$, characterizing the wave distribution outside the slab. To do so, one must solve Maxwell equations inside the slab,

$$\begin{cases} \text{rot}\,\mathbf{E} = ik_0\,\mathbf{H} \\ \text{rot}\,\mathbf{H} = -ik_0\hat{\varepsilon}(z)\,\mathbf{E} \end{cases},$$

with proper boundary conditions – fixed incident wave at the one end and absence of the ingoing (incident) wave at the other end. Here $k_0 = \omega/c$ is the vacuum wave vector, and the dielectric response of plasma is defined by 3×3 tensor operator $\hat{\varepsilon}(z) = \hat{\varepsilon}_0(z,k_x) + \hat{\varepsilon}_1(z,k_x)\,\partial_z + \hat{\varepsilon}_2(z,k_x)\,\partial_z^2 + ...$ which contains derivatives over the z coordinate in case of spatial dispersion.

Now we are ready to introduce the mode-impedance technique. At first step, one must choose the minimum set of field variables that describe a particular problem, and convert Maxwell equations to the following form:

$$\partial_z \mathbf{\Psi} = \hat{M}\,\mathbf{\Psi}\,, \quad (2)$$

where $\hat{M}(z)$ is a matrix of scalar functions of z, it contains no differential operators, and $\mathbf{\Psi}(z)$

is a vector of some of field components. In most cases, the spatial dispersion increases an order of Maxwell equations, thus to preserve the form (2), some derivatives of field components must be also included as components of the $\boldsymbol{\Psi}$ -vector, e.g.

$$\boldsymbol{\Psi} = \left(E_x, H_x, E_y, H_y, E_z, H_z, \partial_z E_x, \partial_z E_y, \partial_z E_z, \partial_z^2 E_x \ldots\right),$$

Here the derivatives are treated as independent field variables what allows to take into account higher dimensionality induced by the spatial dispersion, for example, related to appearance of electrostatic waves. Particular cases of reduction of Maxwell equation to form (2) will be studied in the next sections. In further analysis we assume that the vector field $\boldsymbol{\Psi}(z)$ is continuous; to provide this it is sufficient to require that $\hat{M}(z)$ is finite.

Once Maxwell equation are formulated in the form (2), it may be rewritten as equations for two coupled counterpropagating waves

$$\begin{cases} \partial_z \mathcal{E}^+ = \hat{t}^+ \mathcal{E}^+ + \hat{r}^- \mathcal{E}^- \\ -\partial_z \mathcal{E}^- = \hat{t}^- \mathcal{E}^- + \hat{r}^+ \mathcal{E}^+ \end{cases}. \tag{3}$$

Here $\mathcal{E}^+$ and $\mathcal{E}^-$ describe the waves with definite propagation direction in the external medium outside the slab. Thus we reduce Maxwell equations to equations of coupled waves (3) which should be solved in the range $a < z < b$ with boundary conditions $\mathcal{E}^+(a) = \mathcal{E}^{\text{inc}}, \mathcal{E}^-(b) = 0$. Fortunately the problem may be further reduced to evolutional problem defined only by initial conditions. Let us formally introduce "mode-impedance" matrix that couples forward and backward waves as

. $$\mathcal{E}^-(z) = \hat{R}(z)\mathcal{E}^+(z). \tag{4}$$

Note, that this unknown yet operator may be considered as the reflection matrix of the reduced layer $[z,b]$ for the wave incident from $z < a$. For example, $\hat{R}(a)$ gives the reflection matrix for the whole slab $[a,b]$, and $\hat{R}(b) = 0$ since there is no reflection for waves propagating in the homogeneous external medium. Substituting (4) into equations (3), one may exclude $\mathcal{E}^+$ and obtain the following matrix equation for the impedance operator:

$$-\partial_z \hat{R} = \hat{R}\,\hat{r}^- \hat{R} + \hat{R}\hat{t}^+ + \hat{t}^- \hat{R} + \hat{r}^+, \quad \hat{R}(b) = 0. \tag{5}$$

This is the main result of the paper. In case of one-dimensional inhomogeneity, the evolutional equation is reduced to a set of nonlinear ordinary differential equations for components of the impedance matrix $\hat{R}(z)$, which can be easily integrated numerically for arbitrary distribution of the dielectric tensor in space. The integration should start from the right boundary $z = b$ with zero initial condition. Once $\hat{R}(z)$ is known, the wave field distribution inside the slab may be retrieved as well as the reflected and transmitted waves outside the slab (for a given incident wave). Indeed, the forward wave $\mathcal{E}^{+}(z)$ may be obtained by integrating the following equation from the boundary $z = a$:

$$\partial_z \mathcal{E}^{+} = (\hat{t}^{+} + \hat{r}^{-}\hat{R})\, \mathcal{E}^{+}, \quad \mathcal{E}^{+}(a) = \mathcal{E}^{\text{inc}}. \tag{6}$$

The backward wave $\mathcal{E}^{-}$ may be retrieved using equation (8), then the original wave field inside the slab is obtained from (4).

The proposed formalism has been proved to be very effective in the modeling of wave propagation both in the one- and multi- dimensionally inhomogeneous magnetized plasmas. To provide working examples of application of the proposed technique we consider electromagnetic waves propagating in dense magnetized plasmas in electron cyclotron resonance frequency range. We focus our attention on a problem of linear coupling of the electromagnetic O and X modes in inhomogeneous plasma, excitation of the electrostatic electron Bernstein mode by the X mode and cyclotron damping of these waves (more details see [2]).

3. Conclusion

So, the boundary problem for Maxwell equations is equivalent to two consecutive evolutionary problems given by equations (5) and (6) which may be trivially integrated numerically. In these equations we automatically avoid exponential growth of uncontrolled evanescent modes which may result in instability of numerical integration. However we must pay for such simplification since one of the evolutionary equations becomes nonlinear. It should be stressed here that the only heuristic action in our technique is formulation of a model for the dielectric response (what indeed selects the studied modes) and choice of proper components of the vector $\mathbf{\Psi}$. Although the derived analytical expressions proved to be rather lengthy for particular problems studied below, the numerical integration of the resulted evolutionary equation was very stable even for the most difficult cases involving several very different scale-lengths. In summary, by performing all analytical transformations on a computer we develop a flexible and fast tool for studying full wave problems in complex media.

References

[1] Scott M.R. Invariant Imbedding and its Applications to Ordinary Differential Equations. (London 1973)

[2] Shalashov A.G., Gospodchikov E.D. Plasma Physics and Controlled Fusion, **52(2)**, 025007 (2010)