A new method for calculating the potential distribution in one-dimensional quasi-neutral bounded plasmas

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Abstract

A new method is proposed for calculating the potential distribution $\Phi(z)$ in a one-dimensional quasi neutral bounded plasma. The potential is required to satisfy a plasma equation of the form $n^i\{\Phi\} = \underline{n}^e(\Phi)$, with $\underline{n}^e(\Phi)$ given and $n^i\{\Phi\}$ expressed in terms of trajectory integrals of the ion kinetic equation. The present method is characterized by expanding the inverse profile $\underline{z}(\Phi)$ in a power series in $\sqrt{\Phi}$. As a first application, the potential distribution for a collisionless Tonks-Langmuir discharge with a cold ion source is calculated and compared with the results of [K.-U. Riemann, Phys. Plasmas **13**, 063508 (2006)].

1. Introduction

The plasma-wall transition is one of the most important and most studied problems in plasma physics. One of the most basic bounded-plasma models, a quasi neutral collisionless symmetric discharge with Boltzmann-distributed electrons and a cold ion source provided by electron-impact ionization of cold neutrals, was given by Tonks and Langmuir (TL) [1]. The analytical kinetic solution of this basic TL model was found by Harrison and Thompson [2]. Kamran and Kuhn [4] showed that near the center of the basic TL model [1] the fluid quantities should be Taylor-expanded in $\sqrt{\Phi}$ rather than in Φ.

Here we propose a new method for calculating the potential distribution $\Phi(z)$ in a one-dimensional quasi-neutral bounded plasma (an example of which is the quasi-neutral TL model [1]). The potential is required to satisfy the quasi-neutrality condition ("plasma equation"). The present method is characterized by expanding the inverse profile $\underline{z}(\Phi)$ (with the underline denoting functions of Φ) in a power series in $\sqrt{\Phi}$. As a first application, the potential distribution for a collisionless TL discharge with a *general* electron distribution and the related cold ion source is calculated and compared, for the special case of Boltzmann-distributed electrons, with the results of [3]. It is shown that in the present formulation the problem at hand can be solved with relative ease in the whole quasi neutral region.

2. Model and basic equations

We consider the Tonks-Langmuir (TL) model described in [4] and rewrite the plasma equation (Eq. (11) of [4]) in terms of φ as

$$\underline{n}^i \equiv \frac{1}{\sqrt{2}}\int_0^{\varphi} d\hat{\varphi}_t \frac{\underline{n}^e(\hat{\varphi}_t)}{\sqrt{\varphi-\hat{\varphi}_t}}\frac{d\hat{\underline{x}}_t}{d\hat{\varphi}_t} = \underline{n}^e\left(\sqrt{\varphi}\right), \tag{1}$$

where φ is the normalized electrostatic potential. We will now solve this plasma equation in an analytic-numerical manner in the interval $\varphi_1 \leq \varphi \leq \varphi_2$, with $\varphi_1 = 0$ and φ_2 chosen "sufficiently large", cf. below. We expand $\underline{x}$, $\underline{n}^i$ and $\underline{n}^e$ in power series in $s := \sqrt{\varphi-\varphi_1} = \sqrt{\varphi}$ as

$$\underline{x}(\varphi) \to \underline{x}(s) = \sum_{\nu=0}^{\infty} \underline{\alpha}_\nu s^\nu \simeq \sum_{\nu=0}^{M^\alpha} \underline{\alpha}_\nu s^\nu, \tag{2}$$

with $\underline{\alpha}_0 = 0$ and

$$\underline{\alpha}_\nu := \frac{1}{\nu!}\left[\frac{d^\nu \underline{x}(s)}{ds^\nu}\right]_{s\to 0} \qquad (\nu \geq 1), \tag{3}$$

$$\underline{n}^i(s) = \sum_{\mu=0}^{\infty} \underline{\beta}^i_\mu s^\mu \simeq \sum_{\mu=0}^{M^i} \underline{\beta}^i_\mu s^\mu, \tag{4}$$

and

$$\underline{n}^e(s) = \sum_{\mu=0}^{\infty} \underline{\beta}^e_\mu s^\mu \simeq \sum_{\mu=0}^{M^e} \underline{\beta}^e_\mu s^\mu, \tag{5}$$

where the coefficients

$$\underline{\beta}^e_\mu := \frac{1}{\mu!}\left[\frac{d^\mu \underline{n}^e(s)}{ds^\mu}\right]_{s\to 0} \tag{6}$$

are assumed to be given and the $\underline{\beta}^i_\mu$'s are to be determined from the plasma equation. For the exact solution of the plasma equation, the upper summation indices M^α, M^i and M^e are all infinite, but for our analytic-numerical approximate solution they will have to be assigned appropriate finite values.

By inserting (4) and (5) into the plasma equation (Eq. (1)) we obtain

$$\sum_{\mu=0}^{M^i} \beta^e_\mu s^\mu = \sum_{\mu=0}^{M^e} \beta^e_\mu s^\mu \tag{7}$$

and comparing the coefficients for each μ we obtain

$$\underline{\beta}^i_\mu = \underline{\beta}^e_\mu \qquad \left(\mu = 0, 1, ..., M^{ie}\right), \tag{8}$$

from which the $\underline{\beta}^i_\mu$'s can be determined and which implies that $M^i = M^e =: M^{ie}$.

M	$\underline{x}_s$	φ_s
1	0.9003	1.000
5	0.5750	0.8729
8	0.5722	0.8546
10	0.5721	0.8540

Table 1: Position of the sheath edge and the sheath potential for different values of M

3. Calculating the coefficients $\underline{\alpha}_\nu$

The ion density can be calculated as

$$\begin{aligned} \underline{n}^i(\varphi) \to \underline{n}^i(s) &\simeq \frac{1}{\sqrt{2}} \sum_{\mu=0}^{M^{ie}} \underline{\gamma}_\mu \int_0^s \frac{d\hat{\underline{s}}_t\, \hat{s}_t^\mu}{\sqrt{s^2 - \hat{s}_t^2}} = \sum_{\mu=0}^{M^{ie}} \frac{\underline{\gamma}_\mu}{\sqrt{2}} s^\mu \int_0^1 \frac{d\tau\, \tau^\mu}{\sqrt{1-\tau^2}} \\ &= \sum_{\mu=0}^{M^{ie}} \frac{\underline{\gamma}_\mu I_\mu}{\sqrt{2}} s^\mu = \sum_{\mu=0}^{M^{ie}} \left[\sum_{\nu=0}^{M^\alpha} \underline{B}^e_{\mu\nu} \underline{\alpha}_\nu \right] s^\mu = \sum_{\mu=0}^{M^{ie}-1} \left[\sum_{\nu=1}^{M^\alpha} \underline{B}^e_{\mu\nu} \underline{\alpha}_\nu \right] s^\mu, \end{aligned} \tag{9}$$

with

$$\begin{aligned} &\underline{\gamma}_\mu := \sum_{\nu=0}^{M^\alpha} \underline{\beta}^{e*}_{\mu\nu} \underline{\alpha}_\nu, \quad \underline{\beta}^{e*}_{\mu\nu} := \nu \underline{\beta}^e_{\mu+1-\nu} H(\nu-1) H(\mu+1-\nu), \\ &\tau := \frac{\hat{s}_t}{s}, \quad I_\mu := \int_0^1 \frac{d\tau\, \tau^\mu}{\sqrt{1-\tau^2}}, \quad \underline{B}^e_{\mu\nu} := \frac{\underline{\beta}^{e*}_{\mu\nu} I_\mu}{\sqrt{2}}. \end{aligned} \tag{10}$$

Comparing (9) with (4) we see that $\underline{\beta}^i_\mu = \sum_{\nu=1}^{M^\alpha} \underline{B}^e_{\mu\nu} \underline{\alpha}_\nu$, so that Eqs. (8) read

$$\sum_{\nu=1}^{M^\alpha} \underline{B}^e_{\mu\nu} \underline{\alpha}_\nu = \underline{\beta}^e_\mu \qquad \left(\mu = 0, 1, ..., M^{ie} - 1\right). \tag{11}$$

The expansion coefficients in Eqs. (2) and (5) can be written as coefficient vectors $\vec{\underline{\alpha}} = (\underline{\alpha}_0, \underline{\alpha}_1, ..., \underline{\alpha}_{M^\alpha})$ and $\vec{\underline{\beta}}^e = \left(\underline{\beta}^e_0, \underline{\beta}^e_1, ..., \underline{\beta}^e_{M^{ie}}\right)$, respectively. The $\underline{\alpha}_\nu$'s can then be calculated economically by solving Eqs. (11) in the matrix form

$$\overleftrightarrow{C}\, \vec{\underline{\alpha}} = \vec{\underline{\beta}}^e, \tag{12}$$

where $\overleftrightarrow{C}$ is a $M \times M$ dimensional coefficient matrix, with $M^{ie} = M^\alpha =: M$. According to (11), the matrix equation (12) has the form

$$\begin{pmatrix} \underline{B}^e_{01} & \underline{B}^e_{02} & \cdots & \underline{B}^e_{0M} \\ \underline{B}^e_{11} & \underline{B}^e_{11} & \cdots & \underline{B}^e_{11} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{B}^e_{M-1,1} & \underline{B}^e_{M-1,2} & \cdots & \underline{B}^e_{M-1,M} \end{pmatrix} \cdot \begin{pmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \vdots \\ \underline{\alpha}_M \end{pmatrix} = \begin{pmatrix} \underline{\beta}^e_0 \\ \underline{\beta}^e_1 \\ \vdots \\ \underline{\beta}^e_{M-1} \end{pmatrix}. \tag{13}$$

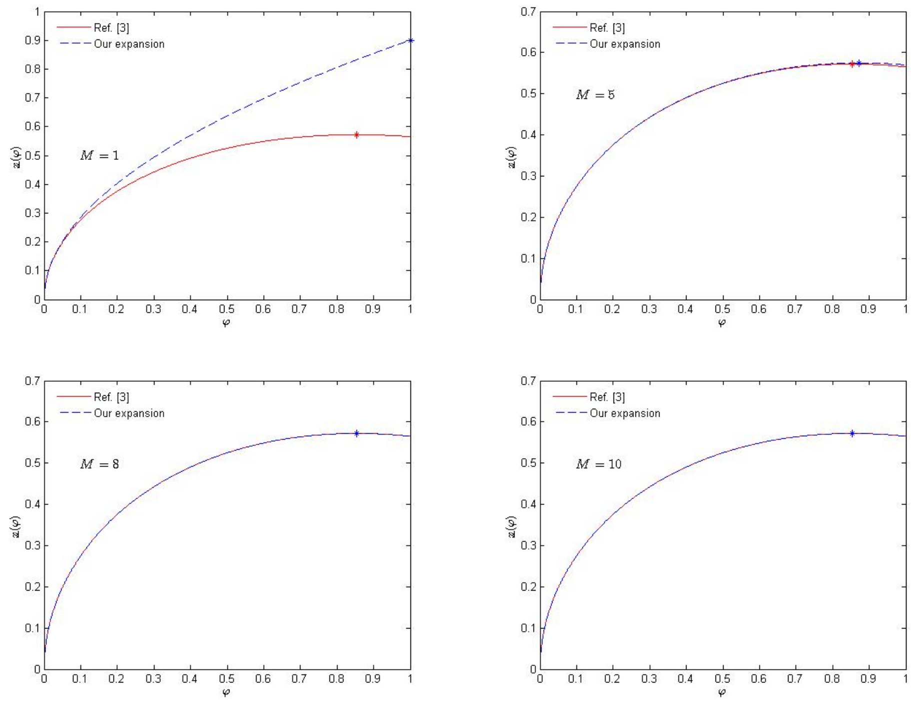

Figure 1: Comparison of the inverse profile $\underline{x}(\varphi)$ calculated with our expansions (2)–(6) with that of Ref. [3].

Figure 1 shows the comparison of the inverse potential profile calculated with our expansions (2)–(6) with that of [3], for the special case of Boltzmann-distributed electrons. The stars denote the maxima of the respective curves, i.e., the sheath-edge singularity (except for $M = 1$, where the "maximum" is actually a supremum). Note that for the $\underline{x}(\varphi)$ profile from [3] we have $\underline{x}_s = 0.5721$ and $\varphi_s = 0.8540$. The values of $\underline{x}_s$ and φ_s for different values of M are given in Table 1, from which we may conclude that for $M \geq 5$ our expansion yields reasonable to excellent results.

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