Nature and dynamics of pseudo- and shear-Alfvén waves overreflection in incompressible MHD shear flows

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Abstract. Wave overreflection - that is a shear flow non-normality induced phenomenon - often determines the dynamics of flow systems. We proposed new route of the overreflection dynamics analysis on an example of incompressible MHD constant shear flow containing pseudo- and shear-Alfvén waves (PAW and SAW): introduced separate/normal variables for each counter propagating wave (that is, Els*ä*sser variables in MHD flows) and, reduced the perturbation equations to two first-order ordinary differential equations for each counter propagating wave. The proposed analysis allows us to separate from each other basic physical processes, to follow their interplay and to gain new insights into the physics of the overreflection. Specifically, our study grasps and describes intrinsic linear coupling of counter propagating waves – the root of the overreflection. It is shown that: (1) PAW with long streamwise wavelength exhibit stronger growth and become more balanced, (2) PAW and SAW branches are not coupled (in the linear limit) with each other, (3) counter propagating SAW are coupled with each other, like counter propagating PAW, (4) the growth and balance degree of SAW are small compared with those of PAW waves. The proposed route is canonical/optimal and is easily applicable to widely discussed cases of the overreflection of spiral-density waves in astrophysical discs and of internal-gravity waves in stably stratified atmospheres.

Introduction. Nonuniform flows occur in atmospheres, solar wind, astrophysical disks, tokamak reactors, etc. Complex dynamics of these systems, in many respects, is determined by their nonuniform kinematics. One of the basic manifestations of flow shear is wave overreflection phenomenon – substantial, self-consistent growth of counter-propagating wave perturbations – that occurs whenever flow has significant shear. Our study, being along the lines of the breakthrough in the understanding of shear flow non-normality induced phenomena performed by the hydrodynamic community in the 1990s (e.g. see [1] and references therein), aims to provide new insight into the physics of the overreflection in 2D incompressible MHD shear flow.

The overreflection phenomenon is usually analyzed on the basis of a linear, second-order ordinary differential (wave) equation (e.g., see [2,3]). This approach describes both counter-propagating waves by one variable and, consequently, possible dynamical processes between these waves (e.g., their coupling) are in fact left out of consideration. We proposed other route of the overreflection investigation – reduce the linear perturbation equations to the set of first-order differential equations for each compound (counter propagating) wave using the nonmodal approach and Els*ä*sser variables (that are normal variables in the shearless limit). The first-order equations separate from each other basic physical processes, make it possible to grasp new physical phenomena – linear coupling of counter propagating PAW – and construct the dynamical picture of the interplay of the basic processes at overreflection. We described in detail self-consistent transient growth of both counter-propagating waves when only one of them exists initially in the flow. The amplification of these (so-called, transmitted and reflected) waves, i.e. *wave overreflection phenomenon*, is determined in many respects by the linear coupling of counter-propagating waves, that is, by shear flow non-normality.

Physical model and equations. Consider a 2D ideal, incompressible MHD fluid flow with constant shear of velocity, $\mathbf{U}_0(Sy,0)$, and uniform magnetic field, $\mathbf{B}_0=(B_0,0)$. The linear dynamic equations of perturbations of the system have the form:

$$(\partial_t+Sy\partial_x)\,v_x+Sv_y=-(1/\rho)\partial_x p\ , \quad \partial_x v_x+\partial_y v_y=0\ , \quad \partial_x b_x+\partial_y b_y=0\ , \tag{1}$$

$$(\partial_t+Sy\partial_x)\,v_y=-(1/\rho)\partial_y p+(B_0/4\pi\rho)\,(\partial_x b_y-\partial_y b_x)\ , \qquad (\partial_t+Sy\partial_x)\,b_y=B_0\partial_x v_y\ , \tag{2}$$

where ρ – mean density; p, $\mathbf{v}$ and $\mathbf{b}$ – pressure, velocity and magnetic field perturbations.

This equations permit the decomposition of perturbed quantities into Kelvin waves, that is the same as spatial Fourier harmonics (SFHs):

$$\Psi(x,y,t)=\Psi(k_x,k_y(t),t)\exp[ik_x x+ik_y(t)y]\ , \tag{3}$$

where $\Psi=\{p,\mathbf{v},\mathbf{b}\}$ and $k_y(t)=k_y(0)-Sk_x t$. Introducing non-dimensional variables: $\tau=St$, $\hat{v}_y=v_y/V_A$, $\hat{b}_y=b_y/B_0$ ($V_A=\sqrt{B_0^2/4\pi\rho}$ – Alfvén velocity), the above system is easily reduced to:

$$\frac{d\hat{v}_y}{d\tau}=2\chi_p(\tau)\hat{v}_y+i\Omega_A\hat{b}_y\ , \quad \frac{d\hat{b}_y}{d\tau}=i\Omega_A\hat{v}_y\ , \quad \chi_p(\tau)\equiv\frac{k_xk_y(\tau)}{k_x^2+k_y^2(\tau)}\ , \quad \Omega_A\equiv\frac{k_xV_A}{S}\ . \tag{4}$$

The proposed route to the description of the overreflection is the following: generally, one have to rewrite the dynamic equations in variables that are normal in the shearless limit. Fortunately, in the considered MHD flow, the normal variables match with the Els*ä*sser variables,

$$Z_p^+=\hat{v}_y-\hat{b}_y\ , \qquad Z_p^-=\hat{v}_y+\hat{b}_y\ . \tag{5}$$

Inserting the Els*ä*sser variables into Eq. (4), we get:

$$\frac{dZ_p^+}{d\tau}=-i\Omega_A Z_p^++\chi_p(\tau)Z_p^++\chi_p(\tau)Z_p^-\ , \quad \frac{dZ_p^-}{d\tau}=i\Omega_A Z_p^-+\chi_p(\tau)Z_p^-+\chi_p(\tau)Z_p^+\ . \tag{6}$$

The dynamics of each of these SFHs is determined by the interplay of three different terms on the right hand side. The second rhs terms of these equations relate to a mechanism of energy exchange between the mean flow and the perturbations. The third rhs terms couple these equations, i.e., couple the counter-propagating waves. χ_p is the time-dependent coupling coefficient of PAW and, in fact, its value determines the strength of the overreflection. The energy density of a SFH (the sum of kinetic and magnetic ones),

$$E(k_x,k_y(\tau),\tau)=1/4\rho V_A^2\left(1+k_y^2(\tau)/k_x^2\right)\left(|Z_p^+|^2+|Z_p^-|^2\right)=E_p^++E_p^-\ , \tag{7}$$

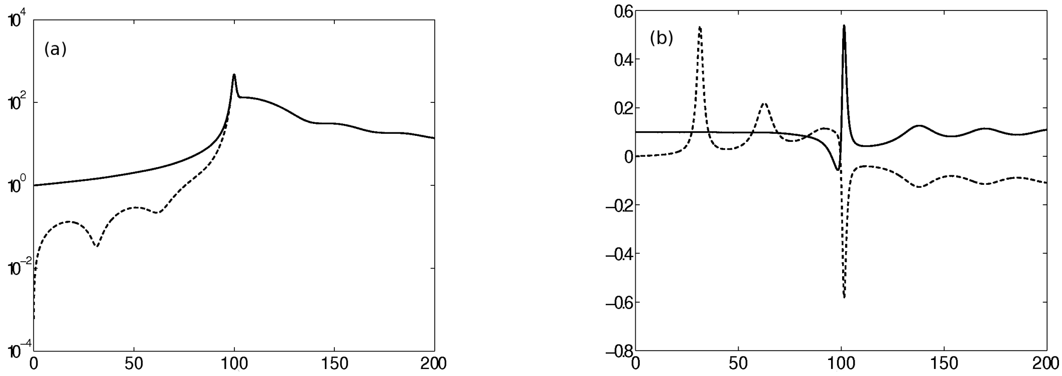

Figure 1: Time variations of: (a) amplitudes, $|Z_p^+|$ (solid) and $|Z_p^-|$ (dashed), in log-linear scaling and (b) instant frequencies, Ω_p^+ (solid) and Ω_p^- (dashed), at $Z_p^+(0) = 1,\ \ Z_p^-(0) = 0$, $k_y(0)/k_x = 100$ and $\Omega_A = 0.1$.

is the sum of energies of the wave SFHs propagating along, E_p^+, and opposite, E_p^-, the X-axis. We introduce polar coordinates, $Z_p^\pm(\tau) = |Z_p^\pm(\tau)|\exp[-i\phi_p^\pm(\tau)]$, and define the imbalance degree of the counter-propagating SFHs, α_p, and "instant frequency", $\Omega_p^\pm(\tau)$:

$$\alpha_p = 1 - \frac{E_p^-}{E_p^+} = 1 - \frac{|Z_p^-|^2}{|Z_p^+|^2}\,, \qquad \Omega_p^\pm(\tau) = \frac{d\phi_p^\pm(\tau)}{d\tau}. \tag{8}$$

Numerical analysis and discussion. We analyze a case of initially unidirectional SFHs: $Z_p^-(0) = 0$. The growth of the perturbations occurs just at $\Omega_A < 1$ and $k_y(0)/k_x > 1$. The intensity of the processes increases with the decrease of Ω_A and increase of $k_y(0)/k_x$. Therefore, we present the results of numerical calculations at $\Omega_A = 0.1; 0.3; 1$ and $k_y(0)/k_x = 100$. Plotted on Fig.1a,b are the time variation of amplitudes, $|Z_p^\pm(\tau)|$, and "instant frequencies", $\Omega_p^\pm(\tau)$. With the help of Eq. (6) and these figures one can trace each stage of the formation of the counter-propagating wave, which represents the overreflection phenomenon. Initially, as $Z_p^-(0) = 0$, only the last rhs term of right Eq. (6) is nonzero. So, the initial amplification and the dynamics of $Z_p^-(\tau)$ is due to the third term $\chi_p Z^+$. Therefore, the positive "instant frequency" of Z_p^+ results in the positive "instant frequency" of Z_p^- (see Fig.1b). The growth of $|Z_p^-|$ is rapid, but algebraic (nonexponential). In the course of evolution, $|Z_p^-|$ becomes almost equal to $|Z_p^+|$ (see Fig.1a), at the same time the influence of the first and second rhs terms of right Eq. (6) become appreciable, behavior of ϕ_p^- and Ω_p^- changes at around $\tau \simeq \tau^* \equiv k_y(0)/k_x = 100$ with Ω_p^- becoming negative and as a result of all these Z_p^- is propagating opposite to Z_p^+. With further increase of time, Ω_p^- tends to $-\Omega_A$ and slightly varies around it. As for the dynamics of Z_p^+, the coupling (the last rhs term of Eq. (6)) somewhat modifies its dynamics in the vicinity of $\tau \simeq \tau^*$, where Z_p^- is already large and χ_p is not small too (while at $\tau \gg \tau^*$, $\chi_p \to 0$).

Fig.2a shows that, starting with a purely unidirectional SFH, $Z_p^-(0) = 0$, the perturbation energy increases and reaches a maximum value at time $\tau \simeq \tau^*$. After that, the SFH undergoes nearly periodic and damping oscillations around some plateau value of the energy. This plateau value decreases with increasing Ω_A. Fig.2b shows that with time the imbalance degree of the SFH decreases, i.e., the energy

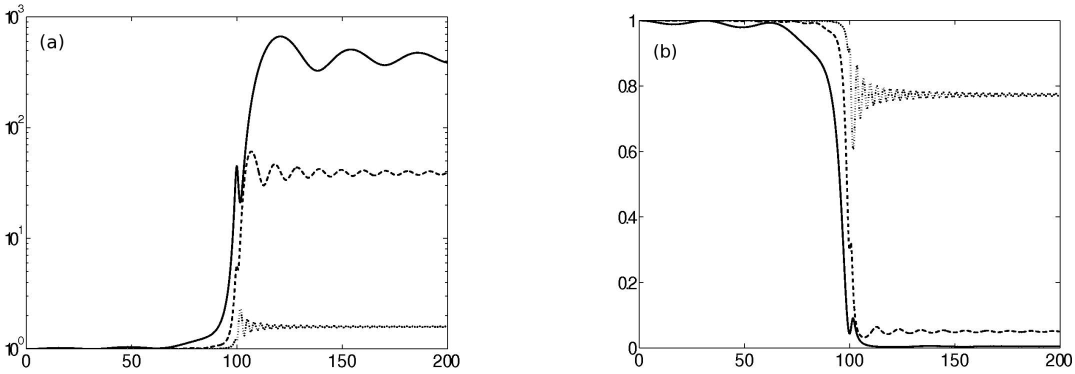

Figure 2: Time variations of: (a) normalized energy of perturbation SFH, $E_p(\tau)/E_p(0)$, and (b) imbalance degree of SFHs, α_p, in log-linear scaling for the same parameters as in Fig.1, but for $\Omega_A = 0.1$ (solid), $\Omega_A = 0.3$ (dashed) and $\Omega_A = 1$ (dotted).

propagating opposite to the X-axis tends to the energy propagating along the X-axis: for the case of $\Omega_A = 1$, the SFH remains imbalanced, for the case of $\Omega_A = 0.3$ the imbalance degree tends to 0.05, and for the case of $\Omega_A = 0.1$, in fact, the SFH becomes balanced with time. Because $\Omega_A \equiv k_x V_A/S$, it is clear that SFHs having smaller k_x (i.e., long streamwise wavelength) exhibit stronger growth and become more balanced.

Generalization of this study to 3D case, where together with PAW, there also exist SAW in the flow, is an easy task. Simple calculations show that equations describing the dynamics of SAW in Els*ä*sser variables are identical to Eq. (6), but with coupling coefficient equal to $\chi_s(\tau) = k_x k_z/k^2(\tau)$, where k_z is the wavenumber along the third Z-axis and $k^2(\tau) = k_x^2 + k_y^2(\tau) + k_z^2$. So, the dynamic equations of the different branches are not couples, i.e., PAW and SAW are not coupled with each other. Due to the change in the coupling coefficient, $\chi_s(\tau)$, the growth of SAW is smaller and the degree of imbalance is high compared with those in the case of PAW. This fact has direct consequence to nonlinear dynamics as nonlinear processes are directly related to interactions between counter propagating waves: the intensity of nonlinear interactions of the waves increases with the decrease of imbalance degree (α_i parameter, where $i = p, s$). Consequently, *ceteris paribus*, the intensity of nonlinear interaction of PAW should prevail the intensity of SAW.

References

P. J. Schmid, Annu. Rev. Fluid Mech **39**, 129 (2007).

P. Goldreich and S. Tremaine, Ap.J. **222**, 850 (1978).

R. S. Lindzen, Pure Appl. Geophys. **126**, 103 (1988).