Asymptotic expansion for stellarator equilibria with a non-planar magnetic axis

A.J. Cerfon[1], F.I. Parra[2] and J.P. Freidberg[1]

[1]*Plasma Science and Fusion Center, MIT, Cambridge MA 02139, USA*

[2]*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3NP, UK*

Because of their fully three-dimensional nature, stellarator equilibria are inherently more complex than tokamak equilibria. In general, they have to be calculated numerically, with computer-intensive solvers. In this paper, we perform an asymptotic analysis based on the smallness of the poloidal magnetic fields compared to the dominant toroidal field in order to simplify the MHD equilibrium equations in stellarators. Our aim is two-fold: 1) accelerate the numerical computation of 3D equilibria; 2) obtain more physical insights into the properties of stellarator equilibria. Our asymptotic analysis generalizes a similar analysis by Greene and Johnson [1] (denoted by GJ) to regimes of interest for modern stellarator experiments such as W7-X, LHD, and HSX.

I. The new stellarator expansion

The stellarator equilibria we consider here have a large toroidal magnetic field B_ϕ, small helical and axisymmetric poloidal magnetic fields $\mathbf{B}_p$, and a small plasma pressure p. We perform an asymptotic analysis of the ideal MHD equilibrium equations relying on the small parameter $\delta \equiv \left|\mathbf{B}_p\right| / B_\phi$. The ordering of the different physical quantities entering in the problem in terms of δ is given in Table 1, and is compared to the GJ ordering [1] . In Table 1, a is the average minor radius, and R_0 is the average major radius. In order to produce a stellarator with a non-planar magnetic axis it is crucial to assume $N \sim 1$, which is the ordering used in the general analysis presented here. This is in contrast to the GJ ordering which assumes that $N \sim 1/\varepsilon$ is large.

At first glance reducing N from order $1/\varepsilon$ to order 1 might seem to be more restrictive. In fact, the non-planar analysis is far more complicated and much less restrictive. It allows for a finite non-planar magnetic axis, finite helical modulations of the flux surfaces, and brings toroidal effects into the calculation earlier, in the same order as helical effects.

Parameter	Symbol	New non-planar scaling	GJ scaling
Plasma beta	$\beta \sim \beta_{toroidal}$	δ	δ^2
Inverse aspect ratio	$\varepsilon \equiv a / R_0$	δ	δ^2
Number of helical periods	N_0	1	$1/\delta^2$
Poloidal helicity	l	1	1
Rotational transform	$\iota / 2\pi$	1	1

Table 1. The non-planar stellarator ordering

Expanding the ideal MHD equilibrium equations to first order in δ with the ordering given in Table 1, we have shown [2] that the (approximate) stellarator equilibrium model is given by the following set of two coupled equations:

$$\begin{aligned} &\left(\varepsilon N_0 \frac{\partial}{\partial \zeta} - \mathbf{e}_\zeta \times \nabla_\perp A_1 \cdot \nabla_\perp\right) \beta_1 = 0 \\ &\left(\varepsilon N_0 \frac{\partial}{\partial \zeta} - \mathbf{e}_\zeta \times \nabla_\perp A_1 \cdot \nabla_\perp\right) \nabla_\perp^2 A_1 = -\frac{\varepsilon}{M^{3/2}} \nabla_\perp \beta_1 \cdot \mathbf{e}_Z \end{aligned} \tag{1}$$

In these equations, the coordinate system is $\left(u, \theta, \zeta\right)$, which is defined as follows:

$$u = \left[M(\zeta)\right]^{1/2} r = \frac{1}{a}\left[M(\zeta)\right]^{1/2}\left[\left(R - R_0\right)^2 + Z^2\right]^{1/2}, \ \theta = \tan^{-1}\left(\frac{Z}{R - R_0}\right), \ \ \zeta = -N_0 \phi \tag{2}$$

where $\left(R, \phi, Z\right)$ is the usual cylindrical coordinate system associated with a toroidal geometry (i.e. for a tokamak, ϕ would be the ignorable coordinate). Also $\beta_1 = 2\mu_0 p_1 / B_0^2$ is the normalized first order pressure (in our ordering, there is no zeroth order pressure), A_1 is a stream function for the poloidal magnetic field, and $M\left(\zeta\right)$ represents the modulation of the toroidal field due to the $l = 0$ mirror field. Specifically, the zeroth order toroidal field is $B_{\zeta 0}(r, \theta, \zeta) = B_{\zeta 0}(\zeta) = B_0 M(\zeta)$, with the average value of M satisfying $\left\langle M \right\rangle = 1$ and B_0 the average toroidal field to lowest order. Once Eq. (1) is solved for the unknown β_1 and A_1, the magnetic field and the current density are easily calculated using the following formulas:

$$\frac{\mathbf{B}}{B_0} = \left[M\left(1 - \frac{\varepsilon}{M^{1/2}}u\cos\theta\right) - \frac{\beta_1}{2M}\right]\mathbf{e}_\zeta + M^{1/2}\nabla_\perp A_1 \times \mathbf{e}_\zeta - \frac{\varepsilon N_0}{2M^{1/2}}\frac{dM}{d\zeta}u\,\mathbf{e}_r$$
$$\frac{\mu_0 a \mathbf{J}_1}{B_0} = -\mathbf{e}_\zeta M \nabla_\perp^2 A_1 - \frac{1}{2M^{1/2}}\nabla_\perp \beta_1 \times \mathbf{e}_\zeta \quad (3)$$

In general, the coupled equations given in Eq. (1) have to be solved numerically. However, we now present semi-analytical solutions to Eq. (1) for vacuum flux surfaces.

II. Vacuum flux surfaces

By vacuum flux surfaces, we mean that $\beta_1 = 0$, and the solution to the second equation in Eq. (1) is $\nabla_\perp^2 A_1 = 0$. Indeed, this means $\mathbf{J}_1 = \mathbf{0}$ according to Eq. (3), as expected. In the following, we assume that we know the shapes and currents in the stellarator coils. The boundary conditions for A_1 are therefore known, so we can solve Laplace's equation for A_1. Thus, A_1 is considered to be a known function. Now, since β_1 is only a function of the toroidal flux $\psi(u,\theta,\zeta)$, when $\beta_1 \to 0$ the first equation in Eq. (1) should be understood as an equation for ψ:

$$\left(\frac{\partial}{\partial\zeta} + \mathbf{e}_\zeta \times \nabla_\perp A \cdot \nabla_\perp\right)\psi = 0 \quad \text{with} \quad \mathrm{A} = -\mathrm{A}_1 / \varepsilon N_0 \quad (4)$$

With A_1 known, Eq. (4) is an advection equation. One might at first think that it can easily be solved using the method of characteristics. The reason this is not so is that in order to follow ψ along the characteristic curves, one first needs to know the initial conditions for ψ, for instance at the initial angle $\zeta = 0$. Finding the appropriate initial conditions such that ψ is single-valued and 2π-periodic in the angle ζ is a very complicated matter.

The approach that we follow instead is to consider magnetic configurations in which the vector potential A has a dominant $l = 2, n = 1$ harmonic, and can be written as follows:

$$A = \alpha u^2 \cos(2\theta + \zeta) + \sum_{l,n} \frac{\alpha_{l,n}}{2} u^{|l|} e^{il\theta + in\zeta} \quad (5)$$

Here, the sum over all l, n excludes the $l = 2, n = 1$ and the $l = -2, n = -1$ which are accounted for in the first term, and $l = 0$, all n, since these contributions appear in the modulation coefficient M. It also excludes $l = \pm 1$, all n. The $l = \pm 1$ fields produce a finite non-planar magnetic axis and are accounted for by a change of coordinate systems [2]. For our analysis we assume that $|\alpha_{l,n}|/\alpha < 1$ for all l,n, and we construct the solution for the unknown flux function by writing an expansion for ψ:

$$\psi = \frac{u^2}{2} + 2A + \psi_1 + \psi_2 + \psi_3 + \ldots \tag{6}$$

The first term represents the normalized flux due to the toroidal field. The second term includes the known contributions due to the dominant $l = 2$ field plus all non-zero harmonics. Their sum would be an exact solution to Eq. (4) if only the $l = 2$ harmonic was non-zero. The remaining terms in Eq. (6) are the expansion corrections, which are treated as small: $\psi_1 / A \sim |\alpha_{l,n}|/\alpha$, $\psi_2 / A \sim (|\alpha_{l,n}|/\alpha)^2$, etc... Inserting this ψ expansion into Eq. (4), we can calculate the corrections ψ_i order by order. At each order, ψ_i is determined to within a free constant, which we choose such that ψ is 2π-periodic in ζ. With this method, we have been able to obtain analytic expressions for ψ_1 and ψ_2 [2], which we can use to analyze vacuum flux surfaces for a variety of configurations (as in Figure 1 for instance).

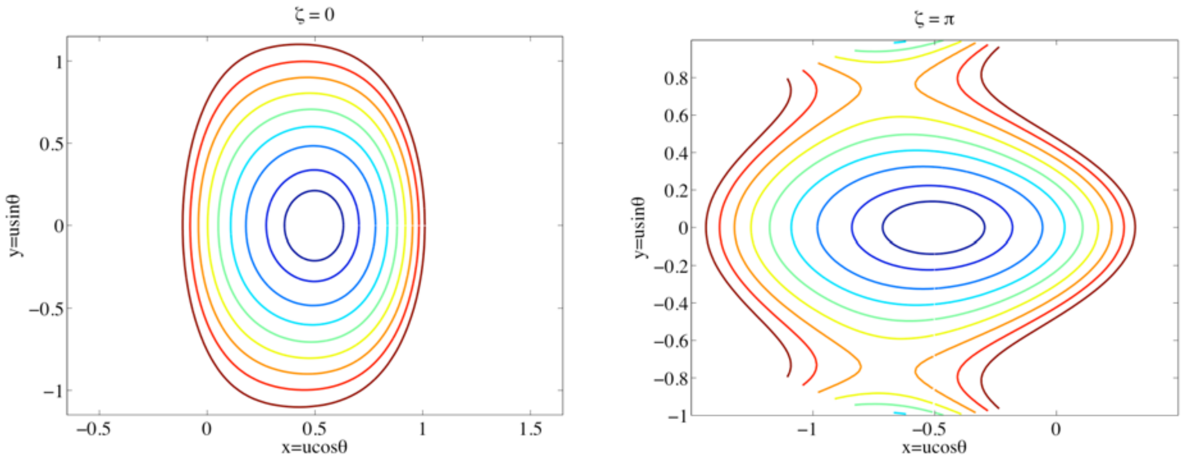

Figure 1. Flux surfaces for an $l = 1,2,3,4; n = 1$ vacuum field in the dominant harmonic expansion. The two cross-sections showed correspond to the angles $\zeta = 0$ and $\zeta = \pi$. $|\alpha_{l,n}|/\alpha = 0.1$ for all harmonics.

In future work we will apply the analytic solutions to actual stellarator experiments, including LHD, W7-X, and HSX. We will also develop an "exact" numerical code to solve Eq. (4) , whose solutions can then be compared to full 3-D MHD codes such as VMEC.

References

[1] J.M. Greene and J.L. Johnson, *Phys. Fluids* **4**, 875 (1961)

[2] A.J. Cerfon, F.I. Parra and J.P. Freidberg, *2011 International Sherwood Fusion Theory Conference, Austin, Texas, May 2-4* (2011)