Numerical and analytical solution of the Lower Hybrid electromagnetic wave equation in one dimensional geometry for propagating and mode converted waves

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Introduction

The use of the lower hybrid (LH) waves is a highly desirable tool for tailoring the plasma current profile in advanced tokamak scenarios. Experiments on LHCD in plasmas with parameters close to those expected in ITER (especially as regarding electron density), have been performed in these last years to master this issue: in FTU [1], in JET [2] and C-Mod [3]. The LH waves are characterized by two modes of propagation called the slow and the fast wave characterized by an Ez/Ex and Ey/Ex wave field polarization respectively. Usually, the lower hybrid waves are launched as slow waves into a tokamak by means of waveguide antennas (grill). Anyway in real plasmas, the non-uniformity of the magnetic field and of the plasma density give rise to critical layers where the slow wave may be converted into the fast wave with a consequent loss of energy. The propagation and the mode conversion of the LH waves is studied analytically and numerically in the following paper by solving the full electromagnetic wave equation (a fourth order ordinary differential equation for the electric field) [4] which is obtained from the Maxwell-Vlasov model.

Wave equation and mode conversion for a LH wave

The relationship between the wave field $\underline{E}$ and the current density $\underline{J}$ in the plasma should be described by solving the Maxwell equation and the Vlasov kinetic equation for the distribution function but some assumption can be made to simplify the analysis. First the fields amplitudes are supposed to be sufficiently small so that $|\underline{E}|^2 << kT$; in this way the kinetic equation can be linearized and the relation between $\underline{E}$ and $\underline{J}$ can be assumed linear as well. Then the plasma medium is supposed to be homogeneous or weakly inhomogeneous, that is the wavelength λ of the radiation is much smaller than the characteristic length L of the non-homogeneities. Finally, if the medium is supposed to be stationary, non dispersive in time and space or slightly dispersive, a Fourier analysis can be carried over. The problem can be further simplified if the wave is propagating in a plane stratified plasma where x is the radial coordinate, y the poloidal and z the toroidal coordinate, with the magnetic field lying along z. Rather than making a Fourier analysis, it can be used a short cut representation when dealing with a single $\underline{k}$ and ω, that is $\underline{E} = \Re\{\hat{\underline{E}}(x)e^{j(k_z z-\omega t)}\} = \Re\{\hat{\underline{E}}(x)e^{j\phi}\}$, having for a LH $k_y = 0$. Besides if the plasma temperature is such that the condition $\lambda >> \rho_L$ is satisfied, the expressions of the hot dielectric tensor can be Taylor expanded. The differential wave equations obtained is the following if the lowest order for T is retained:

$$\begin{aligned} \frac{d^2\hat{E}_y(\bar{x})}{d\bar{x}^2} + \delta_0^{-1}\alpha(\bar{x})\frac{d\hat{E}_z(\bar{x})}{d\bar{x}} + \delta_0^{-2}\beta(\bar{x})\hat{E}_y(\bar{x}) &= 0 \\ \frac{d^2\hat{E}_z(\bar{x})}{d\bar{x}^2} + \delta_0^{-1}\alpha'(\bar{x})\frac{d\hat{E}_y(\bar{x})}{d\bar{x}} + \delta_0^{-2}\beta'(\bar{x})\hat{E}_z(\bar{x}) &= j\delta_0^{-2}\Lambda(\bar{x})\hat{E}_z(\bar{x}) \end{aligned} \tag{1}$$

with $\hat{E}_x(\bar{x}) = -j\frac{D\hat{E}_y(\bar{x})+n_z\delta_0\frac{d\hat{E}_z(\bar{x})}{d\bar{x}}}{n_z^2-S}$

The electric fields $\hat{E}_x$, $\hat{E}_y$ and $\hat{E}_z$ are normalized to the electric fields' intensities whilst the spatial coordinate is such that $\bar{x} = \frac{x}{a}$, where a is the plasma characteristic length. The other quantities are $\alpha = \frac{n_z D}{n_z^2-S}$, $\alpha' = \frac{n_z D}{S}$, $\beta = -\frac{(n_z^2-S)^2-D^2}{n_z^2-S}$, $\beta' = -\frac{(n_z^2-S)^2 P}{S}$, $\delta_0^{-1} = \frac{\omega a}{c}$.

S, D, P are the elements of the cold dielectric tensor whilst $\Lambda = \frac{(n_z^2-S)^2\varepsilon_{zz}^A}{S}$ introduce a dependence on the temperature in the dielectric tensor, where $\varepsilon_{zz}^A = \frac{2\omega_{p\alpha}^2\omega\sqrt{\pi}}{k_z^3 v_{th\alpha}}$ is the anti-hermitian component of the dielectric tensor retained after expansion and $v_{th\alpha}$ the thermal velocities.

Solution for a homogeneous plasma

The Eq. (1) can be easily solved if the dielectric is homogeneous; moreover if the dielectric is cold ($T->0$) the ε_{zz}^A is zero and the analytic expression for the accessible wave is the following:

$$\hat{E}_z(\bar{x}) = \tilde{c}_{11}e^{j\delta_0^{-1}n_{x+}x} + \tilde{c}_{22}e^{-j\delta_0^{-1}n_{x+}x} + \tilde{c}_{33}e^{j\delta_0^{-1}n_{x-}x} + \tilde{c}_{44}e^{-j\delta_0^{-1}n_{x-}x} \tag{2}$$

where the costants $c_1 1...$ are complex quantities and n_{x+}, n_{x-} are the slow and the fast wave refractive indexes.

WKB solution for a non-homogeneous plasma

When the medium is not homogeneous, but $\lambda << L$, and cold, a WKB approximation can be used [4] and neglecting terms greater than δ_0^{-1} it is obtained:

$$\begin{aligned} \hat{E}_y(\bar{x}) = & \frac{w_1}{n_{x+}^{1/2}(n_{x+}^2 - \frac{\beta+\beta'-\alpha\alpha'}{2})^{1/2}} e^{j\delta_0^{-1}\int n_{x+}dx} + \frac{w_2}{n_{x+}^{1/2}(n_{x+}^2 - \frac{\beta+\beta'-\alpha\alpha'}{2})^{1/2}} e^{-j\delta_0^{-1}\int n_{x+}dx} + \\ & \frac{w_3}{n_{x-}^{1/2}(n_{x-}^2 - \frac{\beta+\beta'-\alpha\alpha'}{2})^{1/2}} e^{j\delta_0^{-1}\int n_{x-}dx} + \frac{w_4}{n_{x-}^{1/2}(n_{x-}^2 - \frac{\beta+\beta'-\alpha\alpha'}{2})^{1/2}} e^{-j\delta_0^{-1}\int n_{x-}dx} \end{aligned} \tag{3}$$

The same equation holds for $\hat{E}_z(\bar{x})$. It is worth noting the method remains valid before and after a mode conversion point but it fails on the turning points, where $(\beta+\beta'-\alpha\alpha')^2 - 4\beta\beta' = 0$, as well as in a cut-off where $n_{x\pm} -> 0$.

Numerical solution

A complete analysis can be carried on using a numerical solution. A finite difference method or a Runge-Kutta shooting method is used depending on the boundary condition chosen (Dirichlet or Cauchy). The slow and fast wave propagating in a homogeneous or non-homogeneous plasma are shown in Fig. 1. The figure also depicts a comparison between the numerical solution and the WKB solution, showing a good agreement. In a non-homogeneous plasma, the wave can reach a critical layer depending on the choice of n_z; here a mode conversion happens and part of the slow wave converts to the fast wave and viceversa. Fig. 2 shows how the refractive index n_{x+} and n_{x-} changes as the refractive index n_z varies from about 1.7 to 1.74 . In these cases the waves propagates until they become evanescent as it is depicetd in Fig. 2 for $n_z = 1.719$ (l.h.s.) and $n_z = 1.717$.

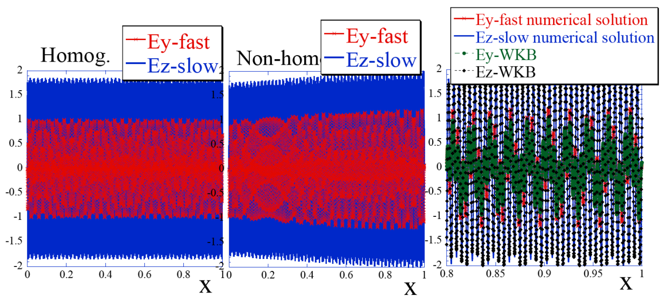

Figure 1: From left: slow and fast wave propagating in a homogeneous, in a non-homogeneous plasma, comparison between a numerical and a WKB solution.

Poynting vector

It is very interesting to calculate the Poynting vector and thus the power carried by the waves. The expression of the Poynting vector in time is well known and very often a complex notation for the electric and magnetic fields is used to calculate the Poynting vector as well. If the waves are periodic in space and time and their amplitude is slowing varying, the final expression can be averaged over time t and space $x-z$.:

$$\underline{P} = \frac{c}{8\pi}\underline{E}(x,z,t)\times\underline{B}(x,z,t) = \frac{c}{8\pi}\Re\{\underline{\hat{E}}(x)\times\underline{\hat{B}}^*(x)e^{2\phi_i}\} \quad (4)$$

where ϕ_i is the imaginary part of the complex phase ϕ.

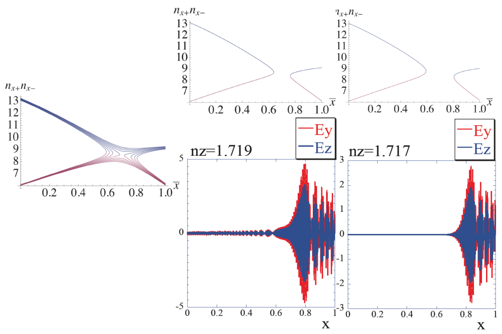

Figure 2: Slow and fast waves encountering a critical layer.

In a slab geometry, the result is the following :

$$\underline{P} = \frac{c}{8\pi}\Re\{< j\frac{c}{\omega}\hat{E}_y\frac{\partial}{\partial x}\hat{E}_y^* + j\frac{c}{\omega}\hat{E}_z\frac{\partial}{\partial x}\hat{E}_z^* - \frac{k_z c}{\omega}\hat{E}_z\hat{E}_x^* >_x\} \quad (5)$$

where the$<>_x$ is the average over the x coordinate. For a homogeneous or non-homogeneous plasma, non dissipative so that $\varepsilon_{zz}^A = 0$, the power carried by the waves is a constant. It is worth noting if the slow wave is excited by the antenna, the coupling between the slow and the fast wave induces the fast wave as well. Anyway most of the power is carried by the slow wave, as Eq.(5) demonstrates, if one compares the contribution coming from the slow ($\hat{E}_z$), the fast ($\hat{E}_y$) and mixed term ($\hat{E}_z\hat{E}_x$)separately. Moreover the analysis of the power carried along the plane stratified plasma during a mode conversion is in progress. As a preliminary study, during a mode conversion part of the wave can tunnel across the critical layer. How much of the wave passes across the critical layer depends on its wavelength λ as compared to the critical layer's length d. If the λ is about equal or greater than the d the wave tunnels across the mode conversion point and most of the power is transferred across the turning point. On the contrary for λ less than the d most of the power is reflected back to the initial point.

Conclusions

The propagation of the LH wave has been analyzed by means of different approaches: analytic, with a WKB method and numeric. A complete analysis can be carried on solving the full wave differential equation for the fields with Dirichlet or Cauchy boundary condition by means of a numeric approach; besides the WKB method confirms the good agreement with the numeric solution when the wave is propagating but it fails on confluence points. The study shows the plasma can be accessible to the wave until it reaches a critical layer where a mode conversion happens. So even if the plasma is inaccessible to a single wave it can be accessible to part of a full a *nz* spectrum, underlying the importance of a deep knowledge of the mode conversion physics. To this purpose, a simple expression for the Poynting vector for a slab geometry has been obtained. Indeed, as a preliminary study, during a mode conversion part of the power is transferred to the companion mode with a consequent lose of efficiency. Thus if most of wave tunnels across the critical layer, it means most of the power is transferred. On the contrary if the wave cannot tunnel, the power is reflected back to the companion mode. The study of this conjecture is in progress.

References

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