δf Monte Carlo computations of parallel conductivity in stellarators *

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Over the last years various methods have been successfully used for the computation of neoclassical transport in non-axisymmetric magnetic field configurations [1]. The evaluation of mono-energetic bootstrap current coefficients has been considerably improved by the utilization of variance reduction techniques [2]. Computations of mono-energetic transport coefficients with this type of methods [3] in many cases are based on pitch-angle scattering collision models, therefore it might be necessary to apply momentum correction techniques in order to compute various plasma quantities of interest. For this purpose the evaluation of the neoclassical conductivity coefficient is essential.

The linearized drift kinetic equation determines the evolution of the first-order guiding center (averaged over the gyro motion) distribution function $\tilde{f}^\sigma$. If a parallel electric field is present and no radial gradients are taken into account this equation takes the form

$$v_\parallel \frac{\partial \tilde{f}^\sigma}{\partial s} - \frac{e v_\parallel E_\parallel}{T} f_M = \nu_d \mathscr{L}_c \tilde{f}^\sigma \,, \tag{1}$$

where $v_\parallel$ is the parallel velocity, s is the distance along the magnetic field line, $\tilde{f}^\sigma = f - f_M$, f is the particle distribution function, f_M is a Maxwellian, e is the particle charge, T is the temperature, $E_\parallel$ is the parallel electric field, ν_d is the deflection collision frequency, $\mathscr{L}_c = 0.5\, \partial/\partial\lambda\,(1-\lambda^2)\partial/\partial\lambda$ is a Lorentz collision operator and σ denotes the sign of $v_\parallel$. Introducing a normalized distribution function $\hat{f}^\sigma$ according to

$$\hat{f}^\sigma = \frac{T\hat{B}}{eE_\parallel}(f_M)^{-1}\tilde{f}^\sigma \,, \tag{2}$$

where $\hat{B} = B/B_0$, B is the module of the magnetic field and B_0 is the reference magnetic field, the drift kinetic equation can be written as

$$\mathscr{L}_D \hat{f}^\sigma \equiv v_\parallel \frac{\partial \hat{f}^\sigma}{\partial s} - \nu_d \mathscr{L}_c \hat{f}^\sigma = \hat{Q}^\sigma \,, \tag{3}$$

where the source is given by $\hat{Q}^\sigma = v_\parallel \hat{B}$. Mono-energetic transport coefficients $D_{II'}$ can be represented as

$$D_{II'} = \alpha_I \alpha_{I'} \left\langle \frac{1}{2}\int_{-1}^{1} \mathrm{d}\lambda\, |\lambda|\, q_I^{-\sigma} \hat{f}_{I'}^\sigma \right\rangle \,, \tag{4}$$

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where $\lambda = v_{\|}/v$ is the pitch angle variable, v is the module of the test particle velocity and

$$\langle A\rangle = \frac{\int \mathrm{d}\vartheta \int \mathrm{d}\varphi \sqrt{g} A}{\int \mathrm{d}\vartheta \int \mathrm{d}\varphi \sqrt{g}} \tag{5}$$

denotes the average over the volume between neighboring flux surfaces, g is the metric determinant of flux coordinates $(\psi, \vartheta, \varphi)$ where ψ is the flux surface label, ϑ and φ are the poloidal toroidal angles of flux coordinates, respectively. For the conductivity coefficient computed in this work the indices I and I' are equal to 3 and omitted in the distribution functions and sources below. The quantity $q_I^{-\sigma}$ in equation (4) is in this case given by $q_3^{-\sigma} = -\sigma\hat{B}$ and the coefficient $\alpha_3 = 1$ which yields for the mono-energetic conductivity coefficient D_{33}

$$D_{33} = -\frac{1}{v}\left\langle \frac{1}{2}\int_{-1}^{1} \mathrm{d}\lambda\, v_{\|} \hat{B} \hat{f}^{\sigma} \right\rangle . \tag{6}$$

In order to apply a Monte Carlo procedure for the solution of the drift kinetic equation it is convenient to re-write (3) in the integral form [2] using a Green's function G defined by $\mathscr{L}_D\, G(t, \mathbf{z}, \mathbf{z}_0) = 0$ and $G(0, \mathbf{z}, \mathbf{z}_0) = \delta(\mathbf{z} - \mathbf{z}_0)/\sqrt{g(\mathbf{z}_0)}$, where $\mathbf{z} = (\vartheta, \varphi, \lambda)$. This Green's function is normalized according to $\int \mathrm{d}^3 z \sqrt{g(\mathbf{z})}\, G(t, \mathbf{z}, \mathbf{z}_0) = 1$. A formal solution to equation (3) is given by

$$\hat{f}^{\sigma}(t, \mathbf{z}) \;=\; \int \mathrm{d}^3 z_0 \sqrt{g(\mathbf{z}_0)} \left(G(t - t_0, \mathbf{z}, \mathbf{z}_0) \hat{f}^{\sigma}(t_0, \mathbf{z}_0) + \int_{t_0}^{t} \mathrm{d}t' G(t - t', \mathbf{z}, \mathbf{z}_0) \hat{Q}^{\sigma}(\mathbf{z}_0) \right) . \tag{7}$$

If a steady state solution is looked for, $\hat{f}^{\sigma}(t, \mathbf{z}) = \hat{f}^{\sigma}(\mathbf{z})$, equation (7) becomes an integral equation for $F^{\sigma}(\mathbf{z}) = \sqrt{g(\mathbf{z})}\, \hat{f}^{\sigma}(\mathbf{z})$ which is given below also in operator form,

$$F^{\sigma}(\mathbf{z}) = \int \mathrm{d}^3 z_0 K(\mathbf{z}, \mathbf{z}_0) F^{\sigma}(\mathbf{z}_0) + Q^{\sigma}(\mathbf{z}) \equiv \mathscr{K} F^{\sigma} + Q^{\sigma}, \tag{8}$$

where $K(\mathbf{z}, \mathbf{z}_0) = \sqrt{g(\mathbf{z})}\, G(\Delta t, \mathbf{z}, \mathbf{z}_0)$, Δt is the integration time step and

$$Q^{\sigma}(\mathbf{z}) = \int \mathrm{d}^3 z_0 \sqrt{g(\mathbf{z})} \sqrt{g(\mathbf{z}_0)} \int_0^{\Delta t} \mathrm{d}t' G(t', \mathbf{z}, \mathbf{z}_0) \hat{Q}^{\sigma}(\mathbf{z}_0) \approx \sqrt{g(\mathbf{z})}\, \Delta t\, \hat{Q}^{\sigma}(\mathbf{z}) . \tag{9}$$

The Monte Carlo operator, $\mathbf{Z}(\Delta t, \mathbf{z}_0)$, is introduced as a random position of a test particle starting at $\mathbf{z}_0$ after a single time step modeled in a standard way [4]. First, the particle pitch is changed randomly in accordance with $\mathscr{L}_C$,

$$\lambda' = \lambda_0 (1 - \Delta_C) + \left(\Delta_C \left(1 - \lambda_0^2\right)\right)^{1/2} \xi, \qquad \Delta_C = \frac{v \Delta t}{l_c}, \tag{10}$$

where ξ is a random number which takes the values ± 1 with equal probabilities, and then an integration step of particle drift equations over the time interval Δt is performed. Thus, the kernel of the integral equation is given by an expectation value

$$K(\mathbf{z}, \mathbf{z}_0) = \overline{\delta\left(\mathbf{z} - \mathbf{Z}(\Delta t, \mathbf{z}_0)\right)} . \tag{11}$$

Equation (11) can be viewed as a definition of the random process $\mathbf{Z}(\Delta t, \mathbf{z}_0)$ via the transition probability density $K(\mathbf{z}, \mathbf{z}_0)$ while the algorithm described in (10) defines a linear approximation in Δt of this random process. At this point, in addition to $\mathbf{Z}(\Delta t, \mathbf{z}_0)$, random numbers $\mathbf{z}_{(k)} \equiv (\vartheta_{(k)}, \varphi_{(k)}, \lambda_{(k)})$ where $k = 0, 1, 2, \ldots$ are introduced via the recurrence relation (13) and the probability density (15). Various overlined quantities below are the expectation values with respect to these random numbers. The solution of (8) by direct iterations can be presented as an expectation value of an integral along the stochastic orbit,

$$F^\sigma = \sum_{k=0}^{\infty} \mathscr{K}^k Q^\sigma = C_0 \sum_{k=0}^{\infty} \overline{w^\sigma_{(0)} \, \delta\left(\mathbf{z} - \mathbf{z}_{(k)}\right)}, \tag{12}$$

$$\mathbf{z}_{(k)} = \mathbf{Z}(\Delta t, \mathbf{z}_{(k-1)}), \tag{13}$$

$$w^\sigma_{(0)} = \Delta t \, \hat{Q}^\sigma(\mathbf{z}_{(0)}) = \Delta t \, v_\parallel(\mathbf{z}_{(0)}) \, \hat{B}(\mathbf{z}_{(0)}), \tag{14}$$

where $C_0 = \int \mathrm{d}^3 z \sqrt{g(\mathbf{z})}$ and the random starting point $\mathbf{z}_{(0)}$ is chosen with the probability density

$$\overline{\delta\left(\mathbf{z} - \mathbf{z}_{(0)}\right)} = C_0^{-1} \sqrt{g(\mathbf{z})}. \tag{15}$$

Substituting the definition of the flux surface average (5) into equation (6) the mono-energetic conductivity coefficient can be written as

$$D_{33} = -\frac{1}{v C_0} \int \mathrm{d}^3 z F^\sigma v_\parallel \hat{B} = -\frac{1}{v} \sum_{k=0}^{\infty} \overline{w^\sigma_{(0)} \, v_\parallel(\mathbf{z}_{(k)}) \, \hat{B}(\mathbf{z}_{(k)})}, \tag{16}$$

where F^σ has been substituted from (12) and integration over $\mathbf{z}$ has been performed in the second equality. The distribution of the test particles at each step remains to be the equilibrium distribution (15). When $k\Delta t$ exceeds a few collision times, the correlation between $\mathbf{z}_{(k)}$ and $w^\sigma_{(0)}$ is lost and such terms in (16) tend to zero. Thus, a finite sum over k is sufficient.

In Figs. 1–6 normalized conductivity coefficients $D_{co} = -D_{33} L_c / l_c$ are plotted versus the collisionality parameter L_c / l_c where $L_c = 2\pi R / \iota$, R is the major radius, ι is the rotational transform and l_c is the mean free path. In these figures are also results from NEO-2, a field line tracing code which computes transport coefficients for zero radial electric fields in arbitrary collisionality regimes [5]. Computations are presented for a variety of non-axisymmetric magnetic field configurations and confinement regimes. Results have been also benchmarked with computations by other methods [1] and stay in good agreement with those results.

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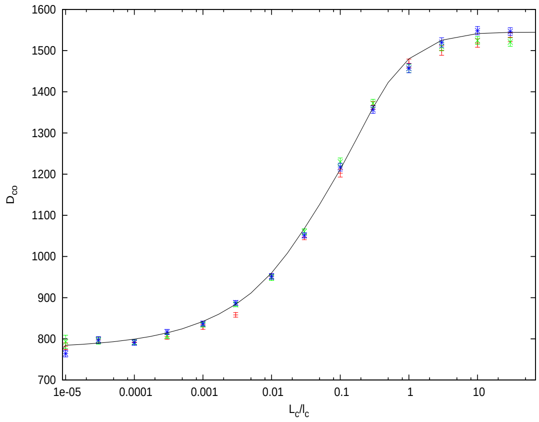

Fig. 1: Normalized conductivity coefficient D_{co} for LHD with R=375cm vs. collisionality parameter L_c/l_c at half plasma radius computed by NEO-2 (line) and NEO-MC (points) for $E_r/(vB) = 0$ (red), $1 \cdot 10^{-4}$ (green), $1 \cdot 10^{-3}$ (blue).

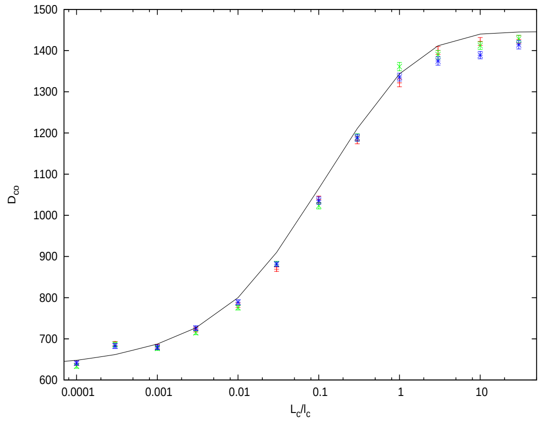

Fig. 2: Normalized conductivity coefficient D_{co} for LHD with R=360cm vs. collisionality parameter L_c/l_c at half plasma radius. Markers and line types are the same as in Fig. 1.

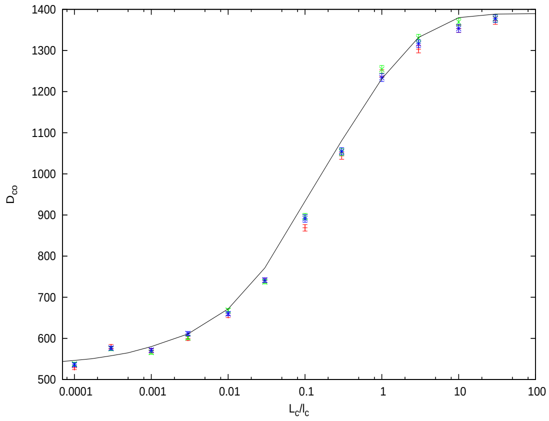

Fig. 3: Normalized conductivity coefficient D_{co} for LHD with R=353cm vs. collisionality parameter L_c/l_c at half plasma radius. Markers and line types are the same as in Fig. 1.

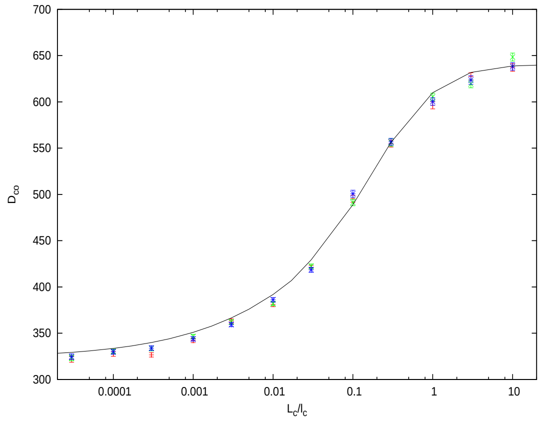

Fig. 4: Normalized conductivity coefficient D_{co} for TJ-II standard configuration vs. collisionality parameter L_c/l_c at half plasma radius. Markers and line types are the same as in Fig. 1.

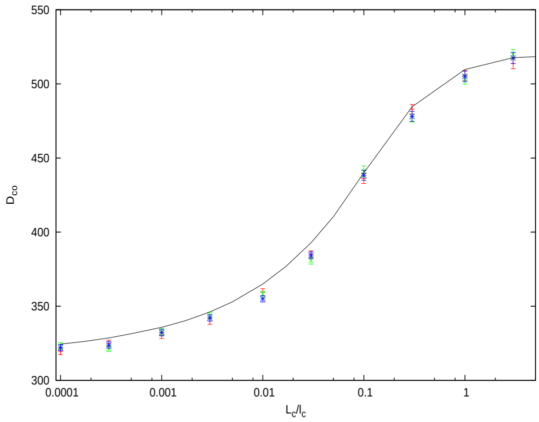

Fig. 5: Normalized conductivity coefficient D_{co} for HSX vs. collisionality parameter L_c/l_c at half plasma radius. Markers and line types are the same as in Fig. 1.

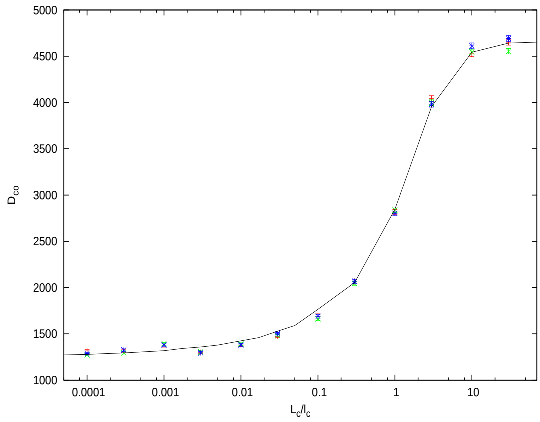

Fig. 6: Normalized conductivity coefficient D_{co} for QIPC vs. collisionality parameter L_c/l_c at half plasma radius. Markers and line types are the same as in Fig. 1.