Second-order electrostatic gyrokinetics in general magnetic geometry and its relevance for toroidal momentum transport in tokamaks

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Introduction

Gyrokinetic theory is the appropriate tool for the kinetic study of microturbulence in fusion plasmas. Let us restrict ourselves to time-independent magnetic fields and collisionless plasmas. Strictly, the set of equations determining the behavior of such a system consists of the Vlasov and Poisson's equations (the latter usually approximated by the quasineutrality equation). However, the time-scale of microturbulence is much longer than the inverse of the gyrofrequency, the time-scale of the gyration of a charged particle around a magnetic field line. It is therefore desirable to eliminate this fast gyromotion in a rigorous way and consequently save much computational time when solving the equations.

Gyrokinetics gives a systematic way to average over the gyromotion (or equivalently over the gyrophase, the degree of freedom associated to it) without losing the effect of non-zero gyroradius based on the smallness of ε, the ratio of the gyroradius and the characteristic length of variation of the magnetic field. In the phase-space Lagrangian formalism the objective is to find a change of variables as an asymptotic series in powers of ε that makes the Lagrangian independent of the gyrophase. This, of course, requires certain assumptions on the size of the quantities entering the theory, that in the so-called low-flow ordering can be summarized as follows:

$$\begin{aligned} &\mathbf{B}(\mathbf{r}) \quad \text{with} \quad \nabla \sim \frac{1}{L} \\ \varphi(\mathbf{r},t) \quad \text{with} \quad \nabla_\perp \sim \frac{1}{\rho}, \quad &\hat{\mathbf{b}}\cdot\nabla \sim \frac{1}{L}, \quad \frac{\partial}{\partial t} \sim \omega \\ &\frac{\omega}{\Omega} \sim \frac{\rho}{L} \sim \frac{Ze\varphi}{Mv_t^2} \sim \varepsilon \ll 1, \end{aligned} \tag{1}$$

where $\mathbf{B}$ is a time-independent magnetic field, L the typical macroscopic length, φ the electrostatic potential, $\rho = v_t/\Omega$, v_t, $\Omega = ZeB/(Mc)$, Ze, and M the typical gyroradius, thermal speed, gyrofrequency, charge and mass of the species of interest, and e and c are the magnitude of the electron charge and the speed of light.

The standard derivation of the gyrokinetic equations in a non-uniform magnetic field is only accurate to first order in ε with the ordering (1). The reason is that two independent expansions are usually performed (see, for example, the review paper [1]). First $\varphi \equiv 0$ is set and an expansion in ε to first order is carried out. Then, the electrostatic potential is switched on an expansion in $\varepsilon_\varphi \sim Ze\varphi/(Mv_t^2)$ is continued up to second order. However, according to (1), $\varepsilon_\varphi \sim \varepsilon$ and the customary derivations miss terms of order $\varepsilon\varepsilon_\varphi$ and ε^2.

In this conference contribution we give the complete result to second order [2], summarized by the full second-order gyrokinetic Hamiltonian (2), (3), (4), (5), clearly exhibiting the new terms obtained from an expansion respecting the ordering (1). The motivation to tackle the second-order computation is found in recent results by Parra and Catto showing that the correct calculation of transport of toroidal angular momentum in tokamaks requires knowledge of second-order pieces of the distribution function and the electrostatic potential [3, 4].

Phase-space Lagrangian and implementation of the gyrokinetic ordering

The phase-space Lagrangian of a charged particle in a time-independent magnetic field reads

$$\mathscr{L}(\mathbf{r},\mathbf{v},\dot{\mathbf{r}},\dot{\mathbf{v}},t) = \left[\frac{Ze}{c}\mathbf{A}(\mathbf{r}) + M\mathbf{v}\right]\cdot\frac{d\mathbf{r}}{dt} - H(\mathbf{r},\mathbf{v},t),$$

where the Hamiltonian is

$$H(\mathbf{r},\mathbf{v},t) = \frac{1}{2}M\mathbf{v}^2 + Ze\varphi(\mathbf{r},t).$$

Introduce dimensionless variables adapted to the gyrokinetic ordering (1),

$$\check{t} = \frac{v_t t}{L},\ \check{\mathbf{r}} = \frac{\mathbf{r}}{L},\ \check{\mathbf{v}} = \frac{\mathbf{v}}{v_t},\ \check{\mathbf{A}} = \frac{\mathbf{A}}{B_0 L},\ \check{\varphi} = \frac{e\varphi}{\lambda\varepsilon T_{e0}},\ \check{H} = \frac{H}{Mv_t^2},$$

giving

$$\check{\mathscr{L}}(\check{\mathbf{r}},\check{\mathbf{v}},\dot{\check{\mathbf{r}}},\dot{\check{\mathbf{v}}},t) = \left[\frac{1}{\varepsilon}\check{\mathbf{A}}(\check{\mathbf{r}}) + \check{\mathbf{v}}\right]\cdot\frac{d\check{\mathbf{r}}}{d\check{t}} - \check{H}(\check{\mathbf{r}},\check{\mathbf{v}},t),$$

with

$$\check{H}(\check{\mathbf{r}},\check{\mathbf{v}},t) = \frac{1}{2}\check{\mathbf{v}}^2 + \Lambda\varepsilon\check{\varphi}.$$

Here, T_{e0} is the characteristic electron temperature. We assume that the characteristic length scale of the electrostatic potencial in the direction perpendicular to the magnetic field is the sound gyroradius, $\rho_s = c_s/\Omega_i$, and the characteristic time scale is L/c_s, being c_s is the sound speed and Ω_i the typical ion gyrofrequency. Concretely,

$$\hat{\mathbf{b}}(\check{\mathbf{r}})\cdot\nabla_{\check{\mathbf{r}}}\check{\varphi} \sim O(1), \quad \nabla_{\check{\mathbf{r}}_\perp}\check{\varphi} \sim O(1/(\lambda\varepsilon)),$$

where

$$\lambda = \rho_s/\rho, \quad \Lambda = ZT_{e0}\lambda/T_0.$$

From now on we work in dimensionless variables but omit hats $\check{}$.

Second-order gyrokinetic phase-space Lagrangian

The program of gyrokinetics can be synthesized by saying that one looks for a transformation from $\{\mathbf{r}, \mathbf{v}\}$ to new phase-space coordinates (*gyrokinetic coordinates*) $\{\mathbf{R}, u, \mu, \theta\}$,

$$(\mathbf{r}, \mathbf{v}) = \mathscr{T}(\mathbf{R}, u, \mu, \theta, t),$$

such that

- $\mathscr{T}$ is expressed as a power series in ε.
- $\mathbf{R}$ is the position of the gyrocenter, and u, μ, θ coincide with the parallel velocity, magnetic moment, and gyrophase to lowest order.
- The transformed phase-space Lagrangian, $\overline{\mathscr{L}}$, is gyrophase-independent.
- μ is an adiabatic invariant.

Doing this up to second-order leads to the following phase-space Lagrangian:

$$\overline{\mathscr{L}} = \left[\frac{1}{\varepsilon}\mathbf{A}(\mathbf{R}) + u\hat{\mathbf{b}}(\mathbf{R}) - \varepsilon\mu\mathbf{K}(\mathbf{R})\right] \cdot \frac{d\mathbf{R}}{dt} - \varepsilon\mu\frac{d\theta}{dt} - \overline{H},$$

with

$$\mathbf{K}(\mathbf{R}) = \frac{1}{2}\hat{\mathbf{b}}(\mathbf{R})\hat{\mathbf{b}}(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \times \hat{\mathbf{b}}(\mathbf{R}) - \nabla_{\mathbf{R}}\hat{\mathbf{e}}_2(\mathbf{R}) \cdot \hat{\mathbf{e}}_1(\mathbf{R}),$$

where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ are unit vectors such that at every point $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$ is an orthonormal set and $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. In order to give the complete and explicit expression for the Hamiltonian to second-order in ε it is useful to define

$$\phi(\mathbf{R}, \mu, \theta, t) := \varphi(\mathbf{R} + \varepsilon\rho(\mathbf{R}, \mu, \theta), t),$$

and the average over θ, denoted by $\langle \dots \rangle$,

$$\langle \dots \rangle := \frac{1}{2\pi}\int_0^{2\pi} (\dots) d\theta.$$

$\widetilde{\phi}$ will stand for the gyrophase-dependent piece of the electrostatic potential, i.e.

$$\tilde{\phi}(\mathbf{R}, \mu, \theta, t) := \phi(\mathbf{R}, \mu, \theta, t) - \langle\phi\rangle(\mathbf{R}, \mu, t),$$

and

$$\tilde{\Phi} := \int^{\theta} \tilde{\phi}\, d\theta,$$

with the choice $\langle\tilde{\Phi}\rangle = 0$. Finally,

$$\overline{H} = \frac{1}{2}u^2 + \mu B + \Lambda\varepsilon\langle\phi\rangle + \varepsilon^2\left(\Lambda^2\Psi_{\phi}^{(2)} + \Lambda\Psi_{\phi B}^{(2)} + \Psi_{B}^{(2)}\right), \quad (2)$$

where

$$\begin{aligned}\Psi_\phi^{(2)} &= \frac{1}{2\lambda^2 B^2}\left\langle \nabla_{(\mathbf{R}_\perp/\lambda\varepsilon)}\widetilde{\Phi}\cdot\left(\hat{\mathbf{b}}\times\nabla_{(\mathbf{R}_\perp/\lambda\varepsilon)}\widetilde{\phi}\right)\right\rangle \\ &- \frac{1}{2\lambda^2 B}\frac{\partial\langle\widetilde{\phi}^2\rangle}{\partial(\mu/\lambda^2)}, \end{aligned} \quad (3)$$

$$\begin{aligned}\Psi_{\phi B}^{(2)} &= -\frac{u}{\lambda B}\left\langle\left(\nabla_{(\mathbf{R}_\perp/\lambda\varepsilon)}\widetilde{\phi}\times\hat{\mathbf{b}}\right)\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}\cdot\rho\right\rangle \\ &- \frac{\mu}{2\lambda B^2}\nabla_{\mathbf{R}}B\cdot\nabla_{(\mathbf{R}_\perp/\lambda\varepsilon)}\langle\phi\rangle - \frac{1}{B}\nabla_{\mathbf{R}}B\cdot\langle\widetilde{\phi}\,\rho\rangle \\ &- \frac{1}{4\lambda B}\left\langle\nabla_{(\mathbf{R}_\perp/\lambda\varepsilon)}\widetilde{\phi}\cdot\left[\rho\rho-(\rho\times\hat{\mathbf{b}})(\rho\times\hat{\mathbf{b}})\right]\cdot\nabla_{\mathbf{R}}B\right\rangle \\ &- \frac{u^2}{\lambda^2 B}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}\cdot\left\langle\frac{\partial\widetilde{\phi}}{\partial(\mu/\lambda^2)}\rho\right\rangle - \frac{u^2}{2\mu B}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}\cdot\langle\widetilde{\phi}\,\rho\rangle \\ &+ \frac{u}{4\lambda^2}\nabla_{\mathbf{R}}\hat{\mathbf{b}}:\left\langle\frac{\partial\widetilde{\phi}}{\partial(\mu/\lambda^2)}\left[\rho(\rho\times\hat{\mathbf{b}})+(\rho\times\hat{\mathbf{b}})\rho\right]\right\rangle \\ &+ \frac{u}{4\mu}\nabla_{\mathbf{R}}\hat{\mathbf{b}}:\left\langle\widetilde{\phi}\left[\rho(\rho\times\hat{\mathbf{b}})+(\rho\times\hat{\mathbf{b}})\rho\right]\right\rangle, \end{aligned} \quad (4)$$

$$\begin{aligned}\Psi_B^{(2)} &= -\frac{3u^2\mu}{2B^2}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}B + \frac{\mu^2}{4B}(\overleftrightarrow{\mathbf{I}}-\hat{\mathbf{b}}\hat{\mathbf{b}}):\nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\mathbf{B}\cdot\hat{\mathbf{b}} \\ &+ \left(\frac{\mu^2}{8}-\frac{u^2\mu}{4B}\right)\nabla_{\mathbf{R}}\hat{\mathbf{b}}:(\nabla_{\mathbf{R}}\hat{\mathbf{b}})^{\mathrm{T}} - \left(\frac{3u^2\mu}{8B}+\frac{\mu^2}{16}\right)(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}})^2 \\ &+ \left(\frac{3u^2\mu}{2B}-\frac{u^4}{2B^2}\right)|\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}}|^2 + \left(\frac{u^2\mu}{8B}-\frac{\mu^2}{16}\right)(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}})^2 \\ &- \frac{3\mu^2}{4B^2}|\nabla_{\mathbf{R}\perp}B|^2 + \frac{u^2\mu}{2B}\nabla_{\mathbf{R}}\hat{\mathbf{b}}:\nabla_{\mathbf{R}}\hat{\mathbf{b}}, \end{aligned} \quad (5)$$

We point out that $\Psi_\phi^{(2)}$ is the term included in the standard derivations of gyrokinetics. It gives an $O(\varepsilon^2)$ contribution because it is quadratic in the fluctuating electrostatic potential. $\Psi_{\phi B}^{(2)}$ (combining geometry and turbulence) and $\Psi_B^{(2)}$ (purely geometrical) are the new terms.

References

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