Orbit-averaged guiding-center Fokker-Planck collision operator with an anisotropic background particle field

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Abstract

The Fokker-Planck equation is routinely used to describe collisional and quasilinear transport processes in magnetized plasmas. It can be transformed using Lie-transform methods to eliminate the fast gyromotion time scale in order to derive an explicit guiding-center Fokker-Planck operator in local guiding-center coordinates [1]. A 3D Fokker-Planck equation for the orbit-averaged distribution function is thus obtained in the low collisionality regime. The expression of an orbit-averaged guiding-center collision operator with an isotropic background particle distribution was recently derived [3].

Here, an orbit-averaged guiding-center collision operator involving an anisotropic local operator is presented. Using the potentials derived by Braams and Karney [2] for the Belaiev-Budker relativistic collision operator, Legendre-polynomial decompositions truncated at first order of the collision coefficients are obtained. A guiding-center Lie transform followed by orbit averaging yields explicit diffusion coefficients with an anisotropic correction to the background particle field. The formalism provides a self-consistent description of neoclassical transport which can also be applied to wave-induced quasilinear transport.

In magnetized toroidal plasmas, the particle gyration Larmor radius ρ is small compared to the characteristic magnetic length L_B. Particle motion can be expressed in terms of the particle energy $\mathscr{E}$ which is a constant of motion, and the adiabatic-invariant magnetic moment μ. Using the small parameter $\varepsilon_B \equiv \rho/L_B \ll 1$, Lie-transform methods are used to derive a guiding-center (GC) Fokker-Planck (FP) collision operator in GC coordinates $(\mathbf{X}, \mathscr{E}, \mu, \varphi)$, where $\mathbf{X}$ denotes the GC position and φ is the gyro-angle [1]. Axisymmetry provides a third invariant: the toroidal canonical momentum P_ϕ which allows us to define a new set of GC coordinates $(\overline{\psi}, \theta, \phi, p, \xi_0, \sigma)$, where the flux-surface label $\overline{\psi} \equiv -(c/e)P_\phi$ and the particle momentum $p \equiv \sqrt{2m\mathscr{E}}$ are constants of motion, the poloidal angle θ is the orbit parameter, the pitch-angle coordinate

$$\xi_0(\overline{\psi}, \mathscr{E}, \mu) \equiv \begin{cases} \sqrt{1 - \mu B_0(\overline{\psi})/\mathscr{E}} & \text{for trapped particles,} \\ \sigma\sqrt{1 - \mu B_0(\overline{\psi})/\mathscr{E}} & \text{for passing particles,} \end{cases} \tag{1}$$

is an adiabatic invariant and $\sigma = \pm 1$. Here, $B_0(\psi)$ is the minimum value of B on the flux-surface ψ.

In the present paper, a FP collision operator with a non-uniform anisotropic background distribution is explicitly derived. In the low-collisionality regime, a 3-D orbit-averaged FP equation is thus obtained in the space of invariants $I^a \equiv (\overline{\psi}, p, \xi_0)$. Simplified orbit-averaged expressions are determined in the thin-orbit width approximation characterized by $\varepsilon_\psi = \varepsilon_B q/\varepsilon \ll 1$, where q is the safety factor and $\varepsilon = r/R$ is the local inverse aspect ratio.

Local anisotropic operator

Starting with a divergence-form local collision operator $C[f_s] \equiv -\partial/\partial\mathbf{p}\cdot(\mathbf{K}f_s - \mathbb{D}\cdot\partial f_s/\partial\mathbf{p})$ between test-particle and background-particle distributions f_s and $f_{s'}$, we perform a Legendre decomposition of $f_{s'}(\mathbf{x},p,\xi) \equiv f_{s'}^{[0]}(\mathbf{x},p) + \xi f_{s'}^{[1]}(\mathbf{x},p) + \dots$ where the local pitch-angle coordinate is $\xi \equiv \sigma\sqrt{1-\left(1-\xi_0^2\right)B/B_0(\overline{\psi})}$. Anisotropic contributions will therefore be obtained from the first-order Legendre term. Using the results from Braams and Karney [2], the local collision coefficients can be rewritten with isotropic and anisotropic first-order Legendre terms,

$$\mathbf{K} = -\left(\nu_l^{[0]} + \xi\nu_l^{[1]}\right)p\mathbf{e}_p - \left(1-\xi^2\right)\nu_t^{[1]}\mathbf{e}_\xi, \tag{2}$$

$$\mathbb{D} = \left(D_l^{[0]} + \xi D_l^{[1]}\right)\mathbf{e}_p\mathbf{e}_p + \left(D_t^{[0]} + \xi D_t^{[1]}\right)(\mathbb{I} - \mathbf{e}_p\mathbf{e}_p) + \frac{1-\xi^2}{p}D_\times^{[1]}\left(\mathbf{e}_p\mathbf{e}_\xi + \mathbf{e}_\xi\mathbf{e}_p\right), \tag{3}$$

where $[\ell]$-terms are solely moment integrals of respective Legendre coefficients $f_{s'}^{[\ell]}$.

GC dynamics

For a GC distribution $F_{gc}(\tau,\overline{\psi},\theta,p,\xi_0)$, the FP equation reads, using the collision time ordering $\varepsilon_\nu \equiv L_B/\lambda_\nu$ in terms of the mean free path λ_ν,

$$\varepsilon_\tau\frac{dF_{gc}}{d\tau} + \dot{\theta}\frac{\partial F_{gc}}{\partial\theta} = \varepsilon_\nu C_{gc}[F_{gc}]. \tag{4}$$

The GC collision operator is expressed in divergence form,

$$C_{gc}[F_{gc}] = -\frac{1}{\mathscr{J}_{gc}}\frac{\partial}{\partial I^a}\left[\mathscr{J}_{gc}\left(K_{gc}^a F - D_{gc}^{ab}\frac{\partial F}{\partial I^b}\right)\right], \tag{5}$$

where $\mathscr{J}_{gc}$ denotes the GC Jacobian and the convection and diffusion GC coefficents respectively read, defining $\langle\dots\rangle_{gc}$ as the gyro-averaging and T_{gc}^{-1} the GC push-forward operator [1],

$$K_{gc}^a = \left\langle \mathsf{T}_{gc}^{-1}\mathbf{K}\cdot\Delta_{gc}^a\right\rangle_{gc}, \qquad D_{gc}^{ab} = \left\langle \Delta_{gc}^a\cdot\mathsf{T}_{gc}^{-1}\mathbb{D}\cdot\Delta_{gc}^b\right\rangle_{gc}. \tag{6}$$

The projection vectors Δ_{gc}^a are the guiding-center Poisson bracket of the particle position with guiding-center coordinates, with known explicit expressions [3].

Orbit-averaged FP equation

The orbit-averaging operation is expressed as $\langle\ldots\rangle_{\mathscr{O}} \equiv \tau_{\mathscr{O}}^{-1}\oint_{\mathscr{O}} d\theta/\dot{\theta}\ldots$, where $\tau_{\mathscr{O}}$ is the orbit time. In low collisionality regimes, the guiding-center distribution $F^{(0)}(\tau,\overline{\psi},p,\xi_0)$ is poloidal-angle independant which allows it to commute with orbit average. We place ourselves in the thin-orbit approximation $\varepsilon_\psi \ll 1$. The orbit-averaged Fokker-Planck equation reads to first order in $\varepsilon_B,\varepsilon_\psi$:

$$\varepsilon_\tau \frac{dF^{(0)}}{d\tau} = -\varepsilon_\nu \frac{1}{\mathscr{J}_{\mathscr{O}}}\frac{\partial}{\partial I^a}\left[\mathscr{J}_{\mathscr{O}}\left(K_{gc}^{a(0)}F^{(0)} - D_{gc}^{ab(0)}\frac{\partial F^{(0)}}{\partial I^b}\right)\right], \tag{7}$$

with $K_{gc}^{a(0)} = \left\langle K_{gc}^{a}\right\rangle_{\mathscr{O}}$ and $D_{gc}^{ab(0)} = \left\langle D_{gc}^{ab}\right\rangle_{\mathscr{O}}$ [3]. The distribution therefore evolves on the collision time scale $\varepsilon_\tau = \varepsilon_\nu$. The Jacobian $\mathscr{J}_{\mathscr{O}} \equiv \tau_{\mathscr{O}} v p^2|\xi_0|/\left(2\pi B_0\left(\overline{\psi}\right)\right)$ is an invariant of the GC motion. The orbit-averaged coefficients read, with isotropic and anisotropic contributions

$$\begin{pmatrix} K_{gc}^{\overline{\psi}(0)} \\ D_{gc}^{p\overline{\psi}(0)} \end{pmatrix} = -\varepsilon_\psi\left[\left\langle \delta\psi \begin{pmatrix} v_l^{[0]} \\ D_l^{[0]}/p \end{pmatrix}\right\rangle_{\mathscr{O}} + \xi_0\left\langle \frac{\xi}{\xi_0}\delta\psi \begin{pmatrix} v_l^{[1]} \\ D_l^{[1]}/p \end{pmatrix}\right\rangle_{\mathscr{O}} \right.$$
$$\left. + \frac{1-\xi_0^2}{\xi_0}\left\langle \frac{\Psi\xi_0}{\xi}\delta\psi \begin{pmatrix} -v_t^{[1]} \\ D_\times^{[1]}/p \end{pmatrix}\right\rangle_{\mathscr{O}}\right], \tag{8}$$

$$\begin{pmatrix} K_{gc}^{p(0)} \\ D_{gc}^{pp(0)} \end{pmatrix} = \left\langle \begin{matrix} v_l^{[0]}p \\ D_l^{[0]} \end{matrix}\right\rangle_{\mathscr{O}} + \xi_0\left\langle \frac{\xi}{\xi_0}\begin{pmatrix} v_l^{[1]}p \\ D_l^{[1]} \end{pmatrix}\right\rangle_{\mathscr{O}} - \varepsilon_B\frac{1-\xi_0^2}{2\xi_0}\left\langle \frac{\Psi\xi_0}{\xi}\lambda_{gc}\begin{pmatrix} v_l^{[1]}p \\ D_l^{[1]} \end{pmatrix}\right\rangle_{\mathscr{O}}, \tag{9}$$

$$\begin{pmatrix} K_{gc}^{\xi_0(0)} \\ D_{gc}^{p\xi_0(0)} \end{pmatrix} = \frac{1-\xi_0^2}{2\xi_0}\left[\left\langle \left(\varepsilon_B\lambda_{gc} + \varepsilon_\psi\overline{\delta\psi}\right)\begin{pmatrix} -v_l^{[0]} \\ D_l^{[0]}/p \end{pmatrix}\right\rangle_{\mathscr{O}} \right.$$
$$\left. + 2\xi_0\left\langle \frac{\xi}{\xi_0}\begin{pmatrix} -v_t^{[1]} \\ D_\times^{[1]}/p \end{pmatrix}\right\rangle_{\mathscr{O}} + \varepsilon_\psi\frac{1-\xi_0^2}{2\xi_0}\left\langle \frac{\Psi\xi_0}{\xi}\overline{\delta\psi}\begin{pmatrix} -v_t^{[1]} \\ D_\times^{[1]}/p \end{pmatrix}\right\rangle_{\mathscr{O}}\right], \tag{10}$$

$$D_{gc}^{\overline{\psi}\overline{\psi}(0)} = 0, \tag{11}$$

$$D_{gc}^{\xi_0\xi_0(0)} = \frac{1-\xi_0^2}{p^2}\left[\left\langle \frac{\xi^2}{\Psi\xi_0^2}\left(1-\varepsilon_B\lambda_{gc}\right)D_t^{[0]}\right\rangle_{\mathscr{O}} + \varepsilon_\psi\frac{1-\xi_0^2}{\xi_0^2}\left\langle \overline{\delta\psi}D_t^{[0]}\right\rangle_{\mathscr{O}} \right.$$
$$+ \xi_0\left\langle \frac{\xi^3}{\Psi\xi_0^3}\left(1-\varepsilon_B\lambda_{gc}\right)D_t^{[1]}\right\rangle_{\mathscr{O}} - \varepsilon_B\frac{1-\xi_0^2}{2\xi_0}\left\langle \frac{\xi}{\xi_0}\lambda_{gc}D_t^{[1]}\right\rangle_{\mathscr{O}}$$
$$\left. + \varepsilon_\psi\frac{1-\xi_0^2}{\xi_0}\left\langle \frac{\xi}{\xi_0}\overline{\delta\psi}D_t^{[1]}\right\rangle_{\mathscr{O}} + \frac{1-\xi_0^2}{\xi_0}\left\langle \frac{\xi}{\xi_0}\left(\varepsilon_B\lambda_{gc} + \varepsilon_\psi\overline{\delta\psi}\right)D_\times^{[1]}\right\rangle_{\mathscr{O}}\right], \tag{12}$$

$$D_{gc}^{\overline{\psi}\xi_0(0)} = -\varepsilon_\psi\frac{1-\xi_0^2}{p^2\xi_0}\left[\left\langle \delta\psi D_t^{[0]}\right\rangle_{\mathscr{O}} + \xi_0\left\langle \frac{\xi}{\xi_0}\delta\psi\left(D_t^{[1]} + D_\times^{[1]}\right)\right\rangle_{\mathscr{O}}\right], \tag{13}$$

where $\nu_l^{[\ell]}, \nu_t^{[\ell]}, D_l^{[\ell]}, D_t^{[\ell]}$ and $D_\times^{[\ell]}$ are functions of GC coordinates (ψ, θ, p, ξ) and $\Psi \equiv B/B_0(\overline{\psi})$. We also have the radial deviation $\delta\psi = \psi - \overline{\psi}$ and the GC vorticity parameter λ_{gc} [1]. This formalism may be applied to other conservative operators and expressions for the ohmic electric field contribution in the FP equation have been similarly derived.

Bootstrap current in the Lorentz limit

In the Lorentz limit $Z \gg 1$ the coefficients of the particle collision operator reduce to $D_t = \nu_{ei} p^2/2$ while ν and D_l can be neglected. The bootstrap current is the flux-surface averaged current density obtained in the presence of collisions. It is evaluated here in the large aspect ratio and thin orbit approximations, with the ordering $\varepsilon \sim \varepsilon_B^{1/2}$ such that $\varepsilon_B \ll \varepsilon, \varepsilon_\psi \ll 1$. Performing an expansion in ε_ψ of the distribution function in the steady-state FP equation, it is shown that $J_b = \left\langle \overline{Zev_\parallel \hat{\mathbf{b}}} \right\rangle_\phi$ reduces [3] to

$$J_b(t,\psi) = -\varepsilon_\psi F_t^{\text{eff}}(\psi) R_p n(\psi) T(\psi) \left[\frac{d\ln n}{d\psi} + \frac{d\ln T}{d\psi} \right]. \tag{14}$$

It can be shown that for the effective trapped particle fraction F_t^{eff}, $\lim_{\varepsilon\to 0} F_t^{\text{eff}}(r) = 1.46\sqrt{\varepsilon} + \mathscr{O}(\varepsilon)$ with circular concentric flux surfaces, which corresponds to previous calculations using drift-kinetic theory [4].

The implementation of this operator in the 3-D Fokker-Planck code LUKE [4] is under way. It will describe neoclassical transport and thus include the bootstrap current consistently with other sources (radio-frequency, ohmic heating) in general current drive calculations.

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