

On the bootstrap current in stellarators and tokamaks

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The aim of this paper is to calculate the collisionless limit of the bootstrap current in a way that makes no distinction between tokamaks and stellarators, and also to consider the correction due to a small but finite collisionality. This correction is proportional to the square root of the collision frequency and tends therefore to be important even in fairly low-collisionality plasmas. We compare the analytical result with a numerical solution of the drift kinetic equation and find excellent agreement in the appropriate collisionality range.

We consider the problem of solving the first-order drift kinetic equation

$$v_{\parallel} \nabla_{\parallel} f_{a1} - \nu_D^a \mathcal{L} f_{a1} = -\mathbf{v}_{da} \cdot \nabla f_{a0} - \frac{e_a v_{\parallel} \nabla_{\parallel} \phi_1}{T_a} f_{a0},$$

where the collision operator has been approximated by pure pitch-angle scattering and the interaction between different species is thus neglected. The notation is standard and follows Ref. [1]. Instead of solving this equation directly, we consider the adjoint equation

$$v_{\parallel} \nabla_{\parallel} g_a + \nu_D^a \mathcal{L}(g_a) = -\frac{v_{\parallel} B}{\langle B^2 \rangle} f_{a0}, \quad (1)$$

from whose solution the bootstrap current can be constructed as

$$\langle J_{a\parallel} B \rangle = \left\langle e_a B \int f_{a1} v_{\parallel} d^3v \right\rangle = -e_a \langle B^2 \rangle \left\langle \int g_a \mathbf{v}_{da} \cdot \nabla \ln f_{a0} d^3v \right\rangle. \quad (2)$$

At low collisionality we may expand $g_a = g_{a0} + g_{a1} + \dots$ in the same way as when treating the banana regime in a tokamak, giving

$$g_{a0} = \frac{\sigma v f_{a0}}{2\nu_D^a} \int_{\lambda}^{1/B_{\max}} \frac{d\lambda'}{\langle \sqrt{1 - \lambda' B} \rangle}. \quad (3)$$

$$g_{a1} = \int_{l_{\max}}^l f_{a0} \left[-\frac{B}{\langle B^2 \rangle} + \frac{1}{B} \frac{\partial}{\partial \lambda} \left(H(B_{\max}^{-1} - \lambda) \frac{\lambda \xi}{\langle \xi \rangle} \right) \right] dl', \quad (4)$$

with $\xi = \sqrt{1 - \lambda B}$. The Heaviside function H appearing in this expression vanishes in the trapped domain, and l_{\max} denotes the position along the field line where the field strength is the largest, $B(l_{\max}) = B_{\max}$. Substituting this expression in Eq. (2) gives the current as a sum of two terms, $\langle J_{a\parallel} B \rangle = J_{a1} + J_{a2}$, corresponding to the two terms in the integrand of Eq. (4). The first one is

$$J_{a1} = \left\langle e_a B \int \frac{\partial f_{a0}}{\partial \psi} 2\pi v^2 dv \int_0^{1/B} \frac{\mathbf{v}_{da} \cdot \nabla \psi}{\xi} d\lambda \int_{l_{\max}}^l B dl' \right\rangle,$$

where ψ is the toroidal flux and

$$\mathbf{v}_{da} \cdot \nabla \psi = \frac{m_a v^2}{e_a} \xi (\mathbf{b} \times \nabla \psi) \cdot \nabla \left(\frac{\xi}{B} \right). \quad (5)$$

It is straightforward to carry out the integrals over v and λ to obtain

$$J_{a1} = p_a A_{1a} \left\langle (\mathbf{B} \times \nabla\psi) \cdot \nabla \left(\frac{1}{B^2} \right) \int_{l_{\max}}^l B dl' \right\rangle,$$

where we have written $A_{1a} = d \ln p_a / d\psi + (e_a / T_a) d\Phi / d\psi$, with $\Phi(\psi)$ the equilibrium electrostatic potential. Writing

$$g_2(l) = B^2 \int_{l_{\max}}^l (\mathbf{b} \times \nabla\psi) \cdot \nabla B^{-2} dl',$$

we obtain $J_{a1} = -p_a A_{1a} \langle g_2 \rangle$. The second term in Eq. (4) makes the following contribution to the current,

$$J_{a2} = -\langle B^2 \rangle \left\langle e_a \int \frac{\partial f_{a0}}{\partial \psi} 2\pi v^2 dv \int_0^{1/B_{\max}} \frac{B \mathbf{v}_{da} \cdot \nabla\psi}{\xi} d\lambda \int_{l_{\max}}^l \frac{\partial}{\partial \lambda} \left(\frac{\lambda \xi}{\langle \xi \rangle} \right) \frac{dl'}{B} \right\rangle,$$

where we substitute the drift velocity (5) and integrate by parts in λ to obtain

$$J_{a1} = \frac{3p_a A_{1a} \langle B^2 \rangle}{4} \int_0^{1/B_{\max}} \frac{\langle g_4 \rangle}{\langle \xi \rangle} \lambda d\lambda,$$

with

$$g_4(\lambda, l) = \xi \int_{l_{\max}}^l (\mathbf{b} \times \nabla\psi) \cdot \nabla \xi^{-1} dl', \quad (\lambda < 1/B_{\max}).$$

The total current of each species is thus

$$\langle J_{a\parallel} B \rangle = J_{a1} + J_{a2} = p_a A_{1a} \left(\frac{3 \langle B^2 \rangle}{4} \int_0^{1/B_{\max}} \frac{\langle g_4 \rangle \lambda d\lambda}{\langle \sqrt{1 - \lambda B} \rangle} - \langle g_2 \rangle \right), \quad (6)$$

as originally found by Shaing and Callen [2] using a very different argument.

The result (6) is valid in the collisionless limit. Hinton and Rosenbluth [3] considered the effect of a small but finite collisionality in the geometry of a large-aspect-ratio tokamak with circular flux surfaces. We extend their calculation to arbitrary axisymmetric geometry and find that the result then also applies to an important class of stellarators. The first correction to the bootstrap current is proportional to the square root of the collisionality and therefore cannot be obtained by simply continuing the expansion of $g_a = g_{a0} + g_{a1} + \dots$ to higher order. Instead, it is found from a boundary-layer analysis of the region around the trapped-passing boundary, $\lambda = 1/B_{\max}$. Because g_{a0} vanishes in the trapped region and is equal to Eq. (3) in the passing region, its derivative $\partial g_{a0} / \partial \lambda$ is discontinuous at the boundary, making the collision operator infinite, since the pitch-angle scattering operator contains two λ -derivatives. Collisions therefore need to be retained in a boundary layer whose width is proportional to the square root of the collisionality. In this layer, Eq. (1) can be replaced by

$$v_{\parallel} \nabla_{\parallel} g_a + \nu_D^a \mathcal{L}(g_a) = 0,$$

with boundary conditions obtained by asymptotic matching to the collisionless solution (3) away from $\lambda = 1/B_{\max}$. Remarkably, this boundary-value problem is identical to one arising in the theory of transport across magnetic islands [4]. The solution, which can be obtained with the Wiener-Hopf method, has the asymptotic behavior

$$g_a(x, y) \rightarrow \pm(c_0 + c_1 x), \quad x \rightarrow -\infty$$

on the circulating side of the boundary layer. Here $c_0/c_1 = \sqrt{2}(\sqrt{2}-1)\zeta(1/2) \simeq -0.855$, ζ denotes the Riemann zeta function, and

$$x(\lambda) = \frac{\lambda B_{\max} - 1}{\sqrt{\nu_{a0}}},$$

$$\nu_{a0} = \frac{2\nu_D^a}{\pi v} \oint \sqrt{1-b(l)} \frac{dl}{b(l)}. \quad (7)$$

with $b = B/B_{\max}$. Hence it is possible to compute the bootstrap current

$$\langle J_{a\parallel} B \rangle = -\frac{Ip_a}{\iota} \left\{ (1 - f_c^{\text{eff}}(v)) \left[A_{1a} + \left(\frac{m_a v^2}{2T_a} - \frac{5}{2} \right) A_{2a} \right] \right\},$$

where $I = RB_\varphi$ and $A_{2a} = d \ln T_a / d\psi$. Here the brackets denote a velocity-space average [1]

$$\{\dots\} = \int (\dots) \frac{m_a v^2}{3T_a} \frac{f_{a0}}{n_a} d^3v,$$

and $f_c^{\text{eff}} = f_c + \delta f_c(v)$ is an effective fraction of circulating particles. Its collisionless value is

$$f_c = \frac{3\langle B^2 \rangle}{4} \int_0^{1/B_{\max}} \frac{\lambda d\lambda}{\langle \sqrt{1-\lambda B} \rangle}, \quad (8)$$

and the correction from the boundary layer is

$$\delta f_c = 0.51 \langle b^2 \rangle \sqrt{\frac{\nu_D^a}{v \langle \sqrt{1-b} \rangle}} \oint \frac{dl}{b} \quad (9)$$

In a large-aspect-ratio tokamak with circular cross section, $B = B_0(1 - \epsilon \cos \theta)$, we obtain

$$f_c^{\text{eff}} \simeq 1 - 1.46\epsilon^{1/2} + \frac{1.35\nu^*{}^{1/2}}{\epsilon^{1/4}}, \quad (10)$$

where $\nu^* = \nu_D^a R / \iota v$.

This calculation applies to an axisymmetric tokamak of arbitrary cross section but appears difficult to generalize to general stellarator geometry. Of course, it immediately applies to quasisymmetric and quasihelically symmetric configurations, since their neo-classical properties are identical (in leading order) to those in tokamaks. However, there is another important but less trivial case amenable to a similar analysis, and this is the case of a perfectly quasi-isodynamic stellarator [5], i.e., a stellarator where the contours of constant $|\mathbf{B}|$ are poloidally closed and the bounce-averaged radial drift vanishes for all particle orbits. The latter property is also referred to as *omnigeneity* in the literature. In such a magnetic field, it can be shown that if $\mathbf{B} = \nabla\psi \times \nabla\alpha$, then the radial excursion (in terms of ψ) for passing particle orbits can be written as [5]

$$\Delta_a = -\frac{\mu_0 J(\psi)}{2\pi} \frac{v_{\parallel}}{\Omega_a} + \frac{\partial}{\partial\alpha} \int_B^{B_{\max}} k \frac{\partial}{\partial B'} \left(\frac{v'_{\parallel}}{\Omega'_a} \right) dB', \quad \lambda < 1/B_{\max}, \quad (11)$$

where $J(\psi)$ is the toroidal current enclosed by the flux surface ψ , and the function $k(\psi, \alpha, B)$ encapsulates all other necessary geometric information of the magnetic field. As shown in Ref. [5], it then follows that the first-order distribution function is equal to

$$f_{a1} = -\Delta_a \frac{\partial f_{a0}}{\partial\psi} + h_a,$$

where as a result of the α -derivative in Eq. (11), which disappears on an orbit average, the equation for h_a is the same as in a tokamak, with the replacement $I(\psi) \rightarrow -\nu\mu_0 J(\psi)/2\pi$. The solution therefore coincides with that in a tokamak, not only in lowest order [5] but also in the trapped-passing boundary layer. In particular, if the flux surface in question does not enclose any net toroidal current, $J(\psi) = 0$, which is the usual situation in a stellarator, then $h_a = 0$ and the bootstrap current vanishes. Thus we conclude that the bootstrap current in a quasi-isodynamic stellarator vanishes to a very high degree of accuracy: not only is there no current in the collisionless limit [5], but the leading correction due to finite collisionality is also absent.

Finally, we verify our results numerically, by using the DKES code [6] for a high-beta-poloidal NSTX equilibrium [7]. Figure 1 shows the bootstrap current coefficient computed numerically for three different radii as well as the result obtained from the analytic representation (8)-(9) of f_c^{eff} and the large-aspect-ratio limit given by the last term in Eq. (10) but using the exact f_c value (8) instead of the first two terms. The DKES results clearly confirm the boundary-layer analysis. With increasing value of the local inverse aspect ratio ϵ , the plateau regime shrinks and is shifted to higher ν^* . The difference between the full analytical result (solid lines) and the large-aspect-ratio limit (dashed lines) increases with ϵ .

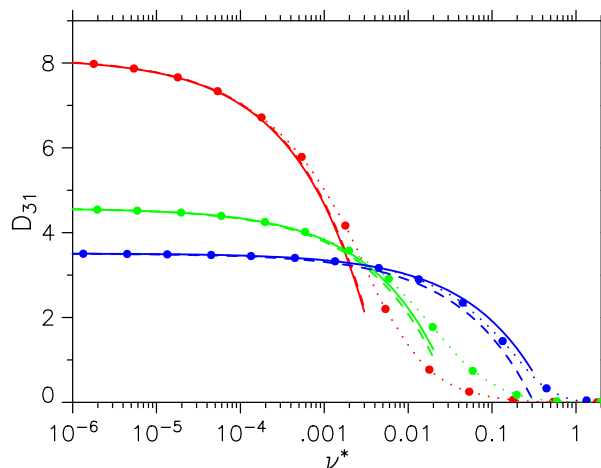


Figure 1: The mono-energetic bootstrap current coefficient vs. collisionality, ν^* , for $\epsilon = 0.0335$, 0.119 and 1.047 (from top to bottom at low ν^*) with DKES results (full circles), the analytical form of f_c^{eff} from Eqs. (8)-(9) (solid lines) and the limit of a large-aspect-ratio tokamak with circular cross section for δf_c (dashed lines). The dotted lines represent interpolations to the DKES data.

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