On Monte Carlo Operators Describing Coulomb Collisions in Toroidal Plasmas

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Modelling of ICRF-heating can be carried out with Monte Carlo codes with operators describing the wave-particle interactions and Coulomb collisions. The collision operator describes the transfer of power to the background species and relaxation towards a local isotropic distribution function. The ion-ion collisions are important for isotropization of the perturbed distribution functions, because they produce a radial electric field in the neoclassical particle transport. In uniform plasma the Coulomb collisions should relax the distribution function towards a Maxwellian with constant density and temperature.

Here a model collision operator, applicable to the banana regime in toroidal plasmas is presented, which has been verified in 2D (in pitch angle and radius). The neoclassical transport is caused by collisional scattering between trapped and passing particles. Since the averaged flux surface location of a trapped particle in axisymmetric plasma is approximately given by the location of its turning points, $\psi_T \approx P_\phi / Ze$, the transport of trapped particles can be calculated from the changes of their canonical angular momentum.

In order to speed up the Monte Carlo simulations, large time steps are preferred. Systematic errors may then occur due to the finite time steps, due to which errors may be increased near boundaries in phase space where the Monte Carlo operators become singular. Furthermore, the Coulomb collision operator should be compatible with neoclassical transport in particular not producing any particle transport caused by the ion-ion collisions.

Model

A generic model, including neoclassical effects in the banana regime is tested, to assess numerical limitations. The orbits are thin, described by (r,ξ), where r is the minor radius and ξ is the pitch angel at the outer midplane. The energy W is constant, only the change in pitch angel is taken into account. Only the trapped particles undergo radial transport as they change their parallel velocity due to pitch angel scattering. For trapped and passing particles the scattering is assumed to take place where the orbits intersect the midplane on the low field side. In neoclassical theory the distribution function deviates slightly from a local Maxwellian such that ion-ion collisions do not give rise to a particle flux.

The change in pitch angel by Coulomb collision in homogeneous plasma is given by

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$$\frac{\partial f}{\partial t}=\frac{1}{2}\frac{\partial}{\partial\xi}\left[\left(1-\xi_n^2\right)\frac{\partial(\gamma f)}{\partial\xi}\right] \qquad (1)$$

The corresponding Euler Monte Carlo scheme becomes,

$$\xi_{n+1}=\xi_n-\xi_n\gamma\Delta t\pm\zeta\sqrt{\left(1-\xi^2\right)\gamma\Delta t} \qquad (2)$$

where $2\upsilon^2\gamma$ is Chandrasekhar's Coulomb diffusion coefficient for pitch angle scattering. If $|\xi_{n+1}|>1$ the particles are reflected at $|\xi_{n+1}|=1$. At the boundary $\psi=\psi_0$ the particles are reflected to obtain steady state solutions; loss of particles will result in a gradual reduction of the total number of particles due to the finite time steps.

In our model only trapped particles undergo radial transport as they change their parallel velocity by collisions. The change in the radial positions of a trapped orbit is obtained from the change in canonical angular momentum

$$\Delta\psi_T=-\frac{mR_x}{Ze}\Delta\upsilon_\phi\approx-\frac{mR_0}{Ze}\Delta\upsilon_\parallel, \qquad (3)$$

where R_x is the position of interaction. The radial position of a passing particle is unchanged. For both trapped and passing particles, the scattering is assumed to take place at the outboard side of the legs. In the banana regime the distribution function of the trapped particles is regarded as symmetric, since the collision frequency is much less than the bounce frequency. Before each collision a random number is used to determine the sign of ξ.

The model operator should not give rise to a net particle transport due to ion-ion collisions. Thus, we have to find a drift term that cancels the particle diffusion due to ion-ion collisions. The Euler Monte Carlo scheme for the change in radius given by

$$\Delta\psi=-C_0\upsilon\left[-\xi_n\gamma\Delta t\pm\zeta\sqrt{\left(1-\xi_n^2\right)\gamma\Delta t}\right] \qquad (4)$$

corresponds to the differential equation

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial\psi}\left[-F\xi_n C_0\gamma\upsilon+\frac{\partial}{\partial\psi}\left(\tfrac{1}{2}\left(1-\xi_n^2\right)C_0{}^2\gamma\upsilon^2F\right)\right]. \qquad (5)$$

To have a steady state solution for $n(\psi)$ we add a drift term $a*$ such that $\partial F/\partial\psi=0$,

$$\frac{\partial F}{\partial t}=\frac{1}{2}\frac{\partial}{\partial\psi}\left[-\frac{F}{J}\frac{\partial}{\partial\psi}\left(C_0J\left(1-\xi_n^2\right)\gamma\right)+2a^*F+\frac{\partial}{\partial\psi}\left(\left(1-\xi_n^2\right)C_0{}^2\gamma\upsilon^2F\right)\right]=0. \qquad (6)$$

To relate F to a local distribution function f we use $F=Jf$, where J is the Jacobian, for a torus $J\propto r^{3/2}$, for which it follows that

$$a^* = \xi_n C_0 \gamma \upsilon - \frac{1}{2J}\frac{\partial}{\partial \psi}\left(\left(1-\xi_n^2\right)C_0{}^2\gamma\upsilon^2 J\right) - \frac{\left(1-\xi_n^2\right)C_0{}^2\gamma\upsilon^2}{2f}\frac{\partial}{\partial\psi} f \quad (7)$$

The corresponding Euler Monte Carlo scheme for radius becomes

$$\psi_{n+1} = \begin{cases} \psi_n + \dfrac{C_0{}^2}{\upsilon^2}\dfrac{1}{2Jn_0(\psi)}\dfrac{\partial}{\partial\psi}\left(\left(1-\xi_n^2\right)\gamma J n_0(\psi)\right) \pm \zeta\sqrt{\dfrac{C_0{}^2\left(1-\xi_n^2\right)\gamma}{\upsilon^2}} & \xi^2 < 2r/R_0 \\ \psi_n & \xi^2 > 2r/R_0 \end{cases} \quad (8)$$

For a circular equilibrium with constant current density we have $r = \sqrt{4\psi/\mu_0 j_0 R_0}$. The convergence is determined by the time step Δt and the radial diffusion by C_0. Fig. 1 demonstrates the comparison of the initial density profile, the density profile after 200 time steps with $\gamma\Delta t = 0.03$ and $C_0 = 3\times10^{-3}$ with the analytical solution. In Fig. 2 the convergence is shown for $C_0 = 3\times10^{-3}$. As C_0 or the time steps increase larger deviation from the analytical solution is obtained. In order to improve the convergence a higher order Monte Carlo method is tested i.e. the Milstein scheme [1].

The Milstein Monte Carlo scheme is given in ξ by

$$\Delta\xi_{n+1} = -\xi_n\gamma\Delta t - \xi\gamma\left(\zeta^2 - 1\right) \pm \zeta\sqrt{\left(1-\xi^2\right)\gamma\Delta t} \quad (9)$$

and in ψ by

$$\psi_{n+1} = \psi_n + \frac{C_0{}^2}{\upsilon^2}\frac{1}{2Jn_0(\psi_n)}\frac{\partial}{\partial\psi}\left(\left(1-\xi_n^2\right)\gamma J n_0(\psi_n)\right) - \xi_n\frac{C_0{}^2\gamma}{\upsilon^2}\left(\zeta^2-1\right) \pm \zeta\sqrt{\frac{C_0{}^2\left(1-\xi_n^2\right)\gamma}{\upsilon^2}} \quad (10)$$

if $\xi^2 < 2r/R_0$ and $\psi_{n+1} = \psi_n$ if $\xi^2 > 2r/R_0$.

Discussion and Conclusion

The Monte Carlo operator, corresponding to pitch angle scattering by ion-ion collisions given by Eqs. (2) and (4), give rise to a radial diffusion of ions producing an asymmetric distribution function. Due to conservation of momentum a corresponding asymmetry would have been found in the field particles, if momentum conservation had been included. In the neoclassical theory it is the drift term arising from collisions with the asymmetric distribution functions that cancels the radial diffusion due to ion-ion collisions. The Monte Carlo operator given by Eqs. (2) and (8) do not give rise to a radial diffusion of ions, it produces an asymmetric distribution function. The asymmetry increases with C_0. It is useful for simulating heating by ion-cyclotron interactions by describing isotropization by pitch angle scattering of ions without giving rise to an unphysical diffusion due to ion-ion collisions. A disadvantage is that, it requires knowledge of the density distribution. A dynamic scheme was tried to cancel the ion-ion diffusion by adding a drift term calculated from the flux. However, the spatial and time variation of the drift term in conjunction with scattering between trapped and passing particles resulted in a large outflux of particles instead of cancelling the transport.

The diffusion coefficient becomes singular: for pitch angle scattering at $\xi = \pm 1$; for radial diffusion at $\psi = 0$ and at the trapped passing boundries. This makes the Monte Carlo modelling difficult. Earlier the singular behaviour near $\xi = \pm 1$ had been studied [2]. The singular behaviour near $\psi = 0$ cause accumulation of counter-passing orbits and a depletion of the co-passing orbits. The reflection at $\psi = \psi_{max}$ gives rise to large errors, in particular, for the Milsten scheme, which does not converge with respect to time steps. Both C_0 and Δt determining the radial transport and are important for convergence.

References

[1]. Peter E Kloeden and Eckhard Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, 1999.

[2]. Q. Mukhtar et al IEEE Plasma Science 38 (2010) 2185.

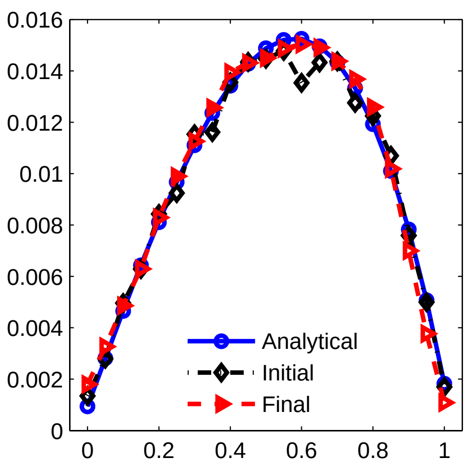

Fig. 1 Comparison of $n(\psi)$ for analytical, initial and steady state after $t = 45$ sec for $\gamma\Delta t = 3\times10^{-2}$ and $C_0 = 3\times10^{-3}$

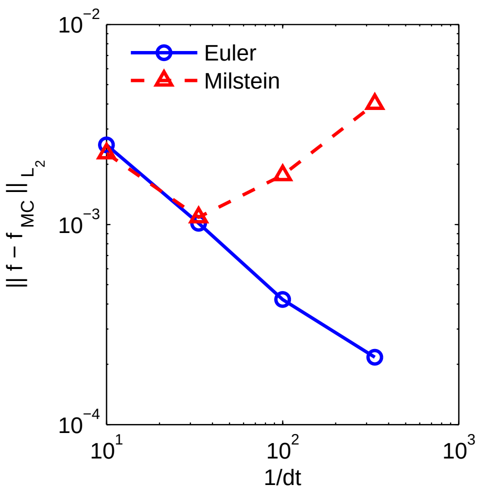

Fig. 2 Convergence study for the Euler and Milsten schemes for $\gamma\Delta t = 3\times10^{-2}$ and $C_0 = 3\times10^{-3}$