Linear drift wave transport in non-Maxwellian plasma

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Experimental observation of the edge turbulence in the fusion devices [1] show that in the Scrape of Layer (SOL) plasma is characterized with non-Gaussian statistics and non-Maxwellian Probability Distribution Function (PDF). It has been recognized that the nature of cross-field transport trough the SOL is dominated by turbulence with a significant ballistic or non-local component and it is not simply a diffusive process [2]. In the present work we study the effect of the non-Maxwellian plasma on anomalous transport using a gyro-kinetic formalism. Here, we consider the application of fractional kinetics to plasma physics. This approach, classical indeed, is new in its application. Our aim is to study the effects of a non-Gaussian statistics on the characteristics of the drift waves in fusion plasmas. We use the solution of the Fokker-Planck equation with a collisional operator consisting of a constant, uniform friction and a stochastic field modeled by alpha-stable statistics represented by a fractional derivative in the velocity space, see Ref. [3]. The solution of the Fokker-Planck equation with fractional velocity derivatives in shearless slab geometry and the stationary state is then plugged into the linearized gyro-kinetic dispersion equation. The dispersion equation is solved numerically and the solutions are presented.

Following the approach used by Barkai [4] we find the Fractional Fokker-Planck Equation (FFPE) with fractional velocity derivatives for shearless slab geometry in the presence of a constant external force as:

$$\frac{\partial F_s}{\partial t} + \mathbf{v}\frac{\partial F_s}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_s}\frac{\partial F_s}{\partial \mathbf{v}} = \nu\frac{\partial}{\partial \mathbf{v}}(\mathbf{v}F_s) + D\frac{\partial^\alpha F_s}{\partial |\mathbf{v}|^\alpha}, \tag{1}$$

where $s(=e,i)$ represents the particle species and $0 \leq \alpha \leq 2$. The diffusion coefficient, D, is related to the damping term,ν, according to a generalized Einstein relation [4]:

$$D = \frac{2^{\alpha-1}T_\alpha \nu}{\Gamma(1+\alpha)m_s^{\alpha-1}}. \tag{2}$$

Here, T_α is a generalized temperature, and taking $\mathbf{F}$ to be the Lorentz force (due to a constant magnetic field and a zero-averaged electric field) acting on the particles of species s with mass m_s and $\Gamma(1+\alpha)$ is the Euler gamma function. To find the solution we make use of the Fourier representation of the above equation as

$$\frac{\partial \mathscr{F}_s}{\partial t} + (-\mathbf{k} + \Omega_s(\mathbf{k}^v \times \hat{b}) + \nu\mathbf{k}^v)\frac{\partial \mathscr{F}_s}{\partial \mathbf{k}^v} = -D|\mathbf{k}^v|^\alpha \mathscr{F}_s, \tag{3}$$

where $\Omega_s = e_s B/m_s c$ is the Larmor frequency of species s, $\hat{b} = \mathbf{B}/B$ is the unit vector in the direction of magnetic field and $\mathscr{F}_s$ is the characteristic function

$$\mathscr{F}_s(\mathbf{k},\mathbf{k}^v;t) = \int\int d\mathbf{r}\, d\mathbf{v} \exp(i\mathbf{k}\cdot\mathbf{r} + i\mathbf{k}^v\cdot\mathbf{v})F_s(\mathbf{r},\mathbf{v};t). \tag{4}$$

Following the method used in Ref. [3, 5] the solution corresponding to the homogenous and steady state system is:

$$F_s(\mathbf{r},\mathbf{v}) = \frac{n_s(\mathbf{r})}{2\pi^{3/2}(\Gamma(1+\alpha))^{-1/2}\sqrt{2V_{T,s}^{\alpha}}}\int \frac{d\mathbf{k}_{\perp}^{v}\,d\mathbf{k}_{\parallel}^{v}}{(2\pi)^{3/2}} e^{-i(\mathbf{k}_{\perp}^{v}\mathbf{v}_{\perp}+\mathbf{k}_{\parallel}^{v}\mathbf{v}_{\parallel})} e^{-\frac{V_{T,s}^{\alpha}}{\Gamma(1+\alpha)\alpha}(|\mathbf{k}_{\perp}^{v}|^{\alpha}+|\mathbf{k}_{\parallel}^{v}|^{\alpha})}. \quad (5)$$

We will now determine the dispersion relation for density gradient driven drift waves. The particle distribution function, averaged over gyro-phase is of the form [6]

$$f_s(\mathbf{r},\mathbf{v}) = F_s(\mathbf{r},\mathbf{v}) + (2\pi)^{-4}\int\int d\mathbf{k}\,d\omega \exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\delta f_{\mathbf{k},\omega}^{s}(\mathbf{v}). \quad (6)$$

We assume that the turbulence is purely electrostatic and neglect magnetic field fluctuations. If the deviation from local equilibrium is not too large, it obeys the well-known linearized gyro-kinetic equation

$$\delta f_{\mathbf{k},\omega}^{s}(v_{\parallel},v_{\perp}) = -\frac{e_s}{T_s}\Big[\frac{\omega-\omega_{*s}}{k_{\parallel}v_{\parallel}-\omega}\Big]J_0(|\Omega_s|^{-1}k_{\perp}v_{\perp})\delta\phi_{\mathbf{k},\omega}\frac{n_s(\mathbf{r})}{2\pi^{3/2}(\Gamma(1+\alpha))^{-1/2}\sqrt{2V_{T,s}^{\alpha}}}\times$$
$$\int \frac{d\mathbf{k}_{\perp}^{v}\,d\mathbf{k}_{\parallel}^{v}}{(2\pi)^{3/2}} e^{-i(\mathbf{k}_{\perp}^{v}\mathbf{v}_{\perp}+\mathbf{k}_{\parallel}^{v}\mathbf{v}_{\parallel})} e^{-\frac{V_{T,s}^{\alpha}}{\Gamma(1+\alpha)\alpha}(|\mathbf{k}_{\perp}^{v}|^{\alpha}+|\mathbf{k}_{\parallel}^{v}|^{\alpha})}. \quad (7)$$

where $\omega_{*s} = \frac{cT_s}{e_sB}k_y \cdot \frac{d\,ln\,n(x)}{dx}$ is the drift wave frequency of species s, and we assumed that the space dependence of F_s is only in the x direction perpendicular to the magnetic field and so is the density gradients. J_0 is the Bessel function of order zero. Here, $v_{\parallel}$ is in the parallel velocity, $v_{\perp} \equiv (v_x^2+v_y^2)^{1/2}$ is the absolute value of the perpendicular velocity, and $v = (v_{\perp}^2+v_{\parallel}^2)^{1/2}$. Inserting the expression for F_s from the equation (5) and rearranging the terms we can solve for the $\delta f_{\mathbf{k},\omega}$. The wave vector perpendicular to magnetic field lines is $k_{\perp} = (k_x^2+k_y^2)^{1/2}$. The gyro-kinetic equation (7) is completed with the Poisson equation for the electric potential. For fluctuations with wave vectors much smaller than the Debye wave vector, the Poisson equation becomes the quasi-neutrality condition

$$\sum_s e_s \delta n_{\mathbf{k},\omega}^{s} = 0. \quad (8)$$

For the density fluctuation therefore we have

$$\delta n_{\mathbf{k},\omega}^{s} = -n_s(\mathbf{r})\frac{e_s}{T_s}\delta\phi_{\mathbf{k},\omega}[M^{ad,s}+M_{\mathbf{k},\omega}^{s}]. \quad (9)$$

Therefore, the dispersion equation as in the Ref. [6] is:

$$M^{ad,e}+M_{\mathbf{k},\omega}^{e} = -M^{ad,i}-M_{\mathbf{k},\omega}^{i} \quad (10)$$

where

$$M^{ad,s} = \int d\mathbf{v}\frac{1}{2\pi^{3/2}(\Gamma(1+\alpha))^{-1/2}\sqrt{2V_{T,s}^{\alpha}}}\int \frac{d\mathbf{k}_{\perp}^{v}\,d\mathbf{k}_{\parallel}^{v}}{(2\pi)^{3/2}} e^{-i(\mathbf{k}_{\perp}^{v}\mathbf{v}_{\perp}+\mathbf{k}_{\parallel}^{v}\mathbf{v}_{\parallel})} e^{-\frac{V_{T,s}^{\alpha}}{\Gamma(1+\alpha)\alpha}(|\mathbf{k}_{\perp}^{v}|^{\alpha}+|\mathbf{k}_{\parallel}^{v}|^{\alpha})}, \quad (11)$$

gives the adiabatic contribution, and

$$M^s_{\mathbf{k},\omega} = \int d\mathbf{v}[\frac{\omega-\omega_{*s}}{k_\parallel v_\parallel - \omega}]J_0(b_s v_\perp/V_{Ts}) \times$$
$$\frac{1}{2\pi^{3/2}(\Gamma(1+\alpha))^{-1/2}\sqrt{2V^\alpha_{T,s}}} \int \frac{d\mathbf{k}^v_\perp d\mathbf{k}^v_\parallel}{(2\pi)^{3/2}} e^{-i(\mathbf{k}^v_\perp \mathbf{v}_\perp + \mathbf{k}^v_\parallel \mathbf{v}_\parallel)} e^{-\frac{V^\alpha_{T,s}}{\Gamma(1+\alpha)\alpha}(|\mathbf{k}^v_\perp|^\alpha + |\mathbf{k}^v_\parallel|^\alpha)}, \tag{12}$$

gives the non-adiabatic contribution. Here, $b_s = k_\perp V_{T,s}/\Omega_s$. If we take $\alpha = 2$ we will recover the dispersion equation for a Maxwellian distribution as in Ref. [6]. The analytical solutions for integrals over $\mathbf{k}^v$ with an arbitrary α in the Equations (11) and (12) requires rather tedious calculations. Instead we consider an infinitesimal deviation of the form $\alpha = 2-\varepsilon$, where $0 \leq \varepsilon \ll 2$ and expand the terms depending on α in the Equations (11) and (12) around $\varepsilon = 0$ as follows

$$\frac{1}{\sqrt{(\Gamma(1+\alpha))^{-1/2}\sqrt{V^\alpha_{T,s}}}} e^{-\frac{V^\alpha_{T,s}}{\Gamma(1+\alpha)\alpha}(|k^v|^\alpha)} = \frac{2^{1/4}e^{-\frac{1}{4}V^2_{T,s}|k^v|^2}}{\sqrt{V_{T,s}}} + \varepsilon\Lambda(k^v) + \mathscr{O}[\varepsilon^2], \tag{13}$$

where

$$\Lambda(k^v) = \frac{e^{-\frac{1}{4}V^2_{T,s}|k^v|^2}}{2^{11/4}\sqrt{V_{T,s}}}\{-3+2\gamma_E - 4V^2_{T,s}|k^v|^2 + 2\gamma_E V^2_{T,s}|k^v|^2$$
$$+2\log[V_{T,s}] + 2V^2_{T,s}\log[V_{T,s}]|k^v|^2 + 2V^2_{T,s}|k^v|^2\log[|k^v|]\}. \tag{14}$$

Here, we have used the Euler-Mascheroni constant $\gamma_E \approx 0.57721$. By using the expansion defined by the expression (13) in Equations (11) and (12), the adiabatic and non-adiabatic part of the dispersion relation $M^{ad,s}$ and $M^s_{\mathbf{k},\omega}$ are as follows

$$M^{ad,s} = 1 + (2\pi\int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp v_\perp \frac{1}{2\sqrt{2}\pi^{3/2}} \int \frac{d\mathbf{k}^v_\perp d\mathbf{k}^v_\parallel}{(2\pi)^{3/2}} e^{-i(\mathbf{k}^v_\perp \mathbf{v}_\perp + \mathbf{k}^v_\parallel \mathbf{v}_\parallel)} \Lambda(k^v_\perp)\Lambda(k^v_\parallel))\varepsilon + \mathscr{O}[\varepsilon]^2$$
$$= 1 + \varepsilon W^{ad,s}. \tag{15}$$

and

$$M^s_{\mathbf{k},\omega} = 2\pi\int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp v_\perp [\frac{\omega-\omega_{*s}}{k_\parallel v_\parallel - \omega}]\Psi_s(b_s v_\perp/V_{Ts}) \frac{1}{(\sqrt{\pi}V_{T,s}(\mathbf{r}))^3} e^{-(v^2_\perp + v^2_\parallel)/V^2_{T,s}(\mathbf{r})} +$$
$$(2\pi\int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp v_\perp [\frac{\omega-\omega_{*s}}{k_\parallel v_\parallel - \omega}]\Psi_s(b_s v_\perp/V_{Ts}) \times$$
$$\frac{1}{2\sqrt{2}\pi^{3/2}} \int \frac{d\mathbf{k}^v_\perp d\mathbf{k}^v_\parallel}{(2\pi)^{3/2}} e^{-i(\mathbf{k}^v_\perp \mathbf{v}_\perp + \mathbf{k}^v_\parallel \mathbf{v}_\parallel)} \Lambda(k^v_\perp)\Lambda(k^v_\parallel))\varepsilon + \mathscr{O}[\varepsilon]^2 = N^s_{\mathbf{k},\omega} + \varepsilon W^s_{\mathbf{k},\omega}. \tag{16}$$

Inserting these relations we may rewrite the dispersion relation (10) in the form

$$(1+N^e_{\mathbf{k},\omega}) + \varepsilon(W^{ad,e} + W^e_{\mathbf{k},\omega}) = -(1+N^i_{\mathbf{k},\omega}) - \varepsilon(W^{ad,i} + W^i_{\mathbf{k},\omega}). \tag{17}$$

The first terms on the right and left hand sides generate the usual contributions to the dispersion equation as in Ref. [6] and the terms proportional to ε generate the non-Maxwellian contributions. Assuming adiabatic electrons and after rearranging the terms in the dispersion equation we get the following relation for ε_i:

$$\varepsilon_i = \frac{-2\xi_i^3 + [\xi_i^3 + 0.5\xi_i - \bar{\omega}_{*,i}\xi_i^2 - 0.5\bar{\omega}_{*,i}]e^{-b_i/2}\mathscr{I}_0(b_i)}{W^{ad,tot}\xi_i^3 + (\bar{\omega}_{*,i}\xi_i^3 - \xi_i^4)Z_\varepsilon(\xi_i)\Gamma_\varepsilon(b_i)}, \tag{18}$$

where $W^{ad,tot} = 2.35\, W^{ad,e} + W^{ad,i}$ and $W^{ad,e,i}$ are the non-Maxwellian contributions of the adiabatic responses for electrons and ions. This relation gives the possible deviation of the equilibrium PDF from the Maxwellian PDF for a given plasma turbulence, i.e $\xi_i = \omega/(|k_\parallel|V_{Ti})$. One has to remember that only positive values of $\mathbf{Re}[\varepsilon]$ are physically meaningful. The dispersion equation is solved numerically. In figure 1, the mode growth rate as a function of ε_i is shown. As the dispersion equation is of 3rd order in $\bar{\omega}$ three possible solutions exist. However we are only interested in the solutions with non-zero imaginary value, $\gamma > 0$ corresponding to unstable situations. It is shown in figure 1 that a deviation of $\varepsilon_i = 0.01$ yield an increase of about 20% in the growth rate. Furthermore, the growth rate increases almost linearly with increasing ε_i and such an increase in the growth rate will lead to a significant increase in the level of anomalous flux.

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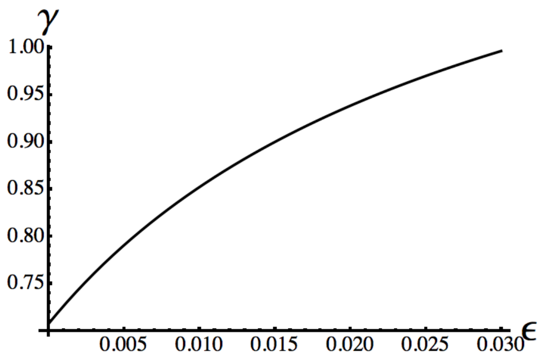

Figure 1: γ as a function of ε. We have assumed $b_i = 0.1$, $k_\parallel = 10^{-3}$ and $\bar{\omega}_{*,i} = -7.1 \times 10^2$ with $d\, ln\, n/dx = 1$.

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