Vlasov-Poisson momentum conservation law in axisymmetric tokamak geometry

N. Tronko [1], A.J. Brizard [2]

[1] *University of Warwick, Center for Fusion Space and Astrophysics, CV4 7AL Coventry, UK*

[2] *Department of Physics, Saint Michael's College, Colchester, VT 05439, USA*

The goal of this communication is to compare the gyrokinetic momentum conservation law recently derived by Scott and Smirnov [1] with the conservation law derived by an application of the Noether method to the variational formulation of Vlasov-Poisson equations [2].

Gyrokinetic angular momentum

We begin with the gyrokinetic angular momentum conservation law

$$\frac{\partial P_\varphi}{\partial t} + \nabla\cdot\Pi_\varphi = 0 \tag{1}$$

for an axisymmetric magnetic field $\mathbf{B} = \nabla\varphi\times\nabla\psi + q(\psi)\nabla\psi\times\nabla\theta$, where the gyrokinetic angular-momentum density

$$P_\varphi = \sum\int F\; p_{gy\,\varphi} d^3p \tag{2}$$

and the canonical gyrokinetic angular momentum flux [2]

$$\Pi_\varphi = \left(\frac{\varepsilon^2}{8\pi}|\mathbf{E}_1|^2\right)\frac{\partial\mathbf{X}}{\partial\varphi} - \frac{\varepsilon^2}{4\pi}\mathbf{E}_1\left(\mathbf{E}_1\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right) + \sum\int F\frac{d_{gy}\mathbf{X}}{dt}p_{gy\,\varphi}d^3p, \tag{3}$$

where $\varepsilon \ll 1$ denotes the ordering parameter associated with the electric-field fluctuations. In an axisymmetic magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$, where $\mathbf{A} = -\psi\nabla\varphi + A_\theta(\psi)\nabla\theta$, the toroidal component of the gyrocenter canonical momentum $\mathbf{p}_{gy} = \frac{e}{c}\mathbf{A} + p_{||}\hat{\mathbf{b}}$ is given by

$$p_{gy\,\varphi} = -\frac{e}{c}\psi + p_{||}\,b_\varphi, \tag{4}$$

where $b_\varphi \equiv \hat{\mathbf{b}}\cdot\partial\mathbf{X}/\partial\varphi$.

Gyrokinetic momentum conservation law in axisymmetric magnetic field

In a following section we give an overview of the principal steps leading from the (1) to Eq. (80) obtained by Scott and Smirnov [1] and its generalization. First we decompose the toroidal angular-momentum density as

$$P_\varphi \equiv -\frac{\psi}{c}\rho + P_{||,\varphi} \tag{5}$$

with the gyrocenter charge density $\rho = \sum\int F\; d^3p$ and the parallel-toroidal gyrocenter momentum density

$$P_{||\varphi} = \left(\sum\int F p_{||}d^3p\right)b_\varphi \tag{6}$$

Next, using the gyrokinetic Poisson equation

$$\frac{\varepsilon}{4\pi}\left(\nabla\cdot\mathbf{E}_1\right) = e\int\langle \mathsf{T}_{gy}^{-1}\delta^3(\mathbf{X}+\boldsymbol{\rho}_{gc}-x)\rangle d^3p,$$

we obtain

$$\frac{\varepsilon^2}{4\pi}\left(\nabla\cdot\mathbf{E}_1\right)\left(\frac{\partial\phi_1}{\partial\varphi}\right) = \left(\sum\int F\frac{\delta H_{gy}}{\delta\phi_1(\mathbf{x})}d^6z\right)\left(\frac{\partial\phi_1}{\partial\varphi}\right) \equiv \sum\int F\frac{\partial H_{gy}}{\partial\varphi}d^3p \tag{7}$$

where we have defined $\partial H_{gy}/\partial\varphi \equiv \varepsilon e\left\langle \mathsf{T}_{gy}^{-1}\partial\phi_{1gc}/\partial\varphi\right\rangle$. We can now use (7) to rearrange the contributions from the Maxwell stress tensor in (3) so that (1) becomes

$$\frac{\partial P_{||\varphi}}{\partial t} + \nabla\cdot\mathbf{Q}_\varphi \;=\; \frac{\psi}{c}\frac{\partial\rho}{\partial t} \;-\; \sum\int F\,\left(\frac{\partial H_{gy}}{\partial\varphi}\right)d^3p, \tag{8}$$

where the gyrocenter angular momentum flux density is

$$\mathbf{Q}_\varphi = \sum\int F\,\frac{d_{gy}\mathbf{X}}{dt}p_{gy\,\varphi}d^3p \tag{9}$$

and the gyrocenter velocity is

$$\frac{d_{gy}\mathbf{X}}{dt} = \frac{p_{||}}{m}\frac{\mathbf{B}^*}{B_{||}^*} + \frac{c\mathbf{b}}{eB_{||}^*}\times\nabla H_{gy}. \tag{10}$$

The next step introduces magnetic-surface averaging

$$\left[\left[\ldots\right]\right] \equiv \frac{1}{\mathscr{V}}\oint(\ldots)\,\mathscr{J}\,d\theta\,d\varphi \tag{11}$$

where $\mathscr{V}(\psi)\equiv\oint\mathscr{J}\,d\theta d\varphi$ is the surface-averaged magnetic-coordinate Jacobian $\mathscr{J}^{-1} = |\nabla\psi\times\nabla\theta\cdot\nabla\varphi|$, which has the general divergence property

$$[[\nabla\cdot\mathbf{C}]] = \frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}\left[\left[\mathbf{C}^\psi\right]\right]\right) \tag{12}$$

We now use the surface-averaged charge conservation law in a combination with partial time derivative gyrocenter quasineutrality condition, which yields the ambipolarity condition on the physical radial current density:

$$\frac{\partial[[\mathscr{P}^\psi]]}{\partial t} + \sum e\left[\left[\int F\frac{d_{gy}\psi}{dt}d^3p\right]\right] = 0, \tag{13}$$

where $\mathscr{P}^\psi$ is the gyrocenter radial polarization. After surface averaging (8), with (13), we obtain Eq. (80) of Ref. [1]:

$$\frac{\partial}{\partial t}\left([[P_{||\varphi}]] + \frac{1}{c}[[\mathscr{P}^\psi]]\right) + \frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}[[Q^\psi_{||\varphi}]]\right) = -\sum e\left[\left[\int F\frac{\partial H_{gy}}{\partial\varphi}d^3p\right]\right] \tag{14}$$

where we have introduced

$$[[Q^\psi_\varphi]] = \sum\left[\left[\int F\,\frac{d_{gy}\psi}{dt}p_{gy\,\varphi}d^3p\right]\right] \equiv [[Q^\psi_{||\,\varphi}]] - \frac{\psi}{c}\sum\left[\left[\int F\,\frac{d_{gy}\psi}{dt}d^3p\right]\right] \tag{15}$$

and

$$[[Q^\psi_{||\,\varphi}]] \equiv \sum\left[\left[\left(\int F\,\frac{d_{gy}\psi}{dt}p_{||}\,d^3p\right)\,b_\varphi\,\right]\right] \tag{16}$$

Let us now highlight some generalizations introduced by the Noether derivation compared with the work of Scott and Smirnov [1].

Generalization I: transport of non-canonical toroidal momentum

The first generalization refers to the definition of the gyrokinetic polarization

$$\mathscr{P} = \sum e \int F \langle \boldsymbol{\rho} \rangle_\varepsilon d^3p - \nabla \cdot \left(\sum \frac{e}{2} \int F \langle \boldsymbol{\rho} \rangle_\varepsilon \langle \boldsymbol{\rho} \rangle_\varepsilon d^3p \right) + \ldots \tag{17}$$

where our derivation [2] takes into account corrections coming from guiding center transformation as well as gyrocenter transformation

$$\langle \boldsymbol{\rho}_\varepsilon \rangle = \langle \boldsymbol{\rho}_{gc1} \rangle + \langle \boldsymbol{\rho}_{gy1} \rangle \equiv -\frac{1}{m\Omega^2} \left(\mu \nabla_\perp B + \frac{p_{||}^2}{m} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right) - \frac{c}{B\Omega} \nabla_\perp \langle \phi_{1gc} \rangle \equiv \frac{\hat{\mathbf{b}}}{\Omega} \times \frac{d_{gy}^{(1)} \mathbf{X}}{dt} \tag{18}$$

where

$$\frac{d_{gy}^{(1)} \mathbf{X}}{dt} = \frac{p_{||}}{m} \frac{\mathbf{B}^*}{B_{||}^*} + \frac{c\hat{\mathbf{b}}}{eB_{||}^*} \times \nabla \left(\mu B + e\varepsilon \langle \phi_1 \rangle \right) \tag{19}$$

is a first order gyrocenter velocity, and $\mathbf{B}^* = \mathbf{B} + (cp_{||}/e) \nabla \times \hat{\mathbf{b}}$ is a magnetic field containing geometrical corrections due to the guiding center transformation. We note here that Scott and Smirnov [1] only consider the gyroangle-averaged gyrocenter displacement $\langle \boldsymbol{\rho}_{gy1} \rangle$. In our case we can rewrite the gyrocenter polarization as

$$\mathscr{P} = \sum \frac{mc}{B} \hat{\mathbf{b}} \times \left[\int F \frac{d_{gy}^{(1)} \mathbf{X}}{dt} d^3p \right] - \nabla \cdot \mathsf{R} \tag{20}$$

where the higher corrections are contained into the second-rank tensor

$$\mathsf{R} \equiv \sum \frac{e}{2} \int F \langle \boldsymbol{\rho}_\varepsilon \boldsymbol{\rho}_\varepsilon \rangle d^3p + \ldots \tag{21}$$

With using the property of axisymmetric magnetic field $\nabla \psi \equiv \mathbf{B} \times \frac{\partial \mathbf{X}}{\partial \varphi}$ we obtain the expression for radial polarization:

$$\mathscr{P}^\psi = \mathscr{P} \cdot \nabla \psi = \sum mc \left[\int F \left(\frac{\partial \mathbf{X}}{\partial \varphi} \cdot \frac{d_{gy}^{(1)} \mathbf{X}}{dt} - \frac{p_{||}}{m} b_\varphi \right) d^3p \right] - \nabla \psi \left(\cdot \nabla \cdot \mathsf{R} \right), \tag{22}$$

where

$$\frac{\partial \mathbf{X}}{\partial \varphi} \cdot \frac{d_{gy}^{(1)} \mathbf{X}}{dt} = \mathscr{R}^2 \frac{d_{gy}^{(1)} \varphi}{dt} \tag{23}$$

with $\mathscr{R} = |\partial \mathbf{X} / \partial \varphi|$. Finally we obtain:

$$\frac{1}{c} \mathscr{P}^\psi + P_{||\,\varphi} = \sum \int F \left(m\mathscr{R}^2 \frac{d_{gy}^{(1)} \varphi}{dt} \right) d^3p - \frac{1}{c} \nabla \psi \left(\cdot \nabla \cdot \mathsf{R} \right) \tag{24}$$

Consideration of full gyrocenter displacement (18) leads to recovery of full toroidal momentum density, consisting of pure guiding center term $-\int F \frac{1}{m\Omega^2} \left(\mu \nabla_\perp B + \frac{p_{||}^2}{m} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right) d^3p$, term proportional to toroidal component of the $\mathbf{E} \times \mathbf{B}$ velocity issued from the gyrocenter correction $-\sum \int F \left(\frac{c}{B\Omega} \nabla_\perp \langle \phi_{1gc} \rangle \right) d^3p$ and parallel toroidal momentum $\sum F (p_{||}/m) b_\varphi d^3p$.

Generalization II: FLR effects

In our work [2] we perform a Taylor expansion of the source term $\partial H_{gy}/\partial\varphi$ in powers of the gyrocenter displacement $\boldsymbol{\rho}_\varepsilon$ defined in (18):

$$\frac{\partial H_{gy}}{\partial\varphi} = \varepsilon e\left(\frac{\partial\phi_1}{\partial\varphi} + \langle\boldsymbol{\rho}_\varepsilon\rangle\cdot\nabla\frac{\partial\phi_1}{\partial\varphi} + \frac{1}{2}\langle\boldsymbol{\rho}_\varepsilon\boldsymbol{\rho}_\varepsilon\rangle : \nabla\nabla\frac{\partial\phi_1}{\partial\varphi} + \ldots\right) \tag{25}$$

Next we obtain the gyrocenter Vlasov-momentum equation with using the gyrocenter polarization (17)

$$\sum\int F\,\frac{\partial H_{gy}}{\partial\varphi}d^3p = \nabla\cdot\left[\varepsilon\,\mathscr{P}\,\frac{\partial\phi_1}{\partial\varphi} + \varepsilon\Big(\sum\frac{e}{2}\int F\langle\boldsymbol{\rho}_\varepsilon\boldsymbol{\rho}_\varepsilon\rangle d^3p\Big)\cdot\nabla\frac{\partial\phi_1}{\partial\varphi} + \ldots\right] \tag{26}$$

Meanwhile Scott and Smirnov considered only the long-wavelength limit of the equation below, with $\mathscr{P} \simeq -\,\varepsilon(mnc^2/B^2)\nabla_\perp\phi_1$ and all the higher order effects are omitted.

Finally we obtain gyrokinetic parallel-toroidal momentum conservation law:

$$\frac{\partial}{\partial t}\left([[P_{\|\varphi}]] + \frac{1}{c}[[\mathscr{P}^\psi]]\right) + \frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}[[Q^\psi_{\|\varphi}]]\right)$$
$$+\frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left[\mathscr{V}\left(\varepsilon\left[\left[\mathscr{P}^\psi\frac{\partial\phi_1}{\partial\varphi} + \mathbf{R}^\psi\cdot\nabla\frac{\partial\phi_1}{\partial\varphi} + \ldots\right]\right]\right)\right] = 0 \tag{27}$$

where

$$\mathbf{R}^\psi \equiv \nabla\psi\cdot\left(\sum\frac{e}{2}\int F\langle\boldsymbol{\rho}_\varepsilon\boldsymbol{\rho}_\varepsilon\rangle d^3p\right) \equiv \mathbf{R}^\psi_0 + \varepsilon\mathbf{R}^\psi_{1gy} + \ldots \tag{28}$$

includes the guiding-center FLR corrections related to the lower-order guiding-center displacement $\boldsymbol{\rho}_0$, as well as higher-order gyrocenter corrections $\varepsilon\mathbf{R}^\psi_{1gy}$.

In this communication two principal generalizations of the gyrokinetic momentum conservation law derived by Scott and Smirnov [1] have been identified. Consideration of full displacement $\boldsymbol{\rho}_\varepsilon$ in the definition of the gyrocenter polarization leads to recovery of toroidal momentum density with $P_{\|\,\varphi} + c^{-1}\mathscr{P}^\psi$ into the transport equation (14) as well as identification of the FLR corrections to the residual stress tensor $\mathbf{R}$. The full Noether derivation of the gyrokinetic momentum conservation law in a general geometry is presented in [2].

References

[1] B. Scott and J. Smirnov, Physics of Plasmas **17**, 112302 (2010)

[2] A.J. Brizard and N. Tronko, submitted to Physics of Plasmas