Self consistent thermal wave model description of the transverse dynamics for relativistic charged particle beams in magnetoactive plasmas

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The plasma wake field (PWF) excitation [1] is one of the efficient mechanisms to generate, in a plasma, ultra-intense electric and magnetic fields. To produce these fields, a relativistic electron/positron beam is launched into the plasma. Due to the very strong nonlinear and collective beam-plasma interaction, the beam becomes the driver of a large amplitude plasma wave that follows the beam with almost its own speed (*plasma wake*) and carries out both longitudinal and transverse electric fields (*plasma wake fields*). If the beam length is much greater than the plasma wavelength (*long beam limit*), the entire beam experiences the effects of the wake fields that itself has produced (*self interaction*).

We study the self interaction of an electron/positron beam travelling in a magnetoactive, collisionless, cold plasma in the overdense regime, i.e., $n_0 \gg n_b$, (n_0 and n_b being the unperturbed plasma and beam densities, respectively). We assume that: a strong constant and uniform external magnetic field acts along the z-axis, $B_0 = B_0\hat{\mathbf{e}}_z$; the electron/positron beam is travelling along the z-axis, i.e. $\mathbf{v}_b = \beta c\hat{\mathbf{e}}_z$ ($\beta \simeq 1$); the ions are at rest to form a background of positive charge. Moreover, we consider a fluid model of the *beam-plasma* system, characterized by the electron fluid velocity, $\mathbf{u}(\mathbf{r},t)$, the electron plasma and beam number densities, $n(\mathbf{r},t)$ and $\rho_b(\mathbf{r},t)$, respectively, the electron plasma and the beam current densities, $-en\mathbf{u}$ and $\beta q\rho_b c\,\hat{\mathbf{e}}_z$, respectively, where $q=-e$ and $q=e$ for electrons and positrons, respectively. We ignore the longitudinal beam dynamics and concentrate our investigation on the transverse effects only. We express the total electric and magnetic fields of the beam-plasma system in terms of the four-potential $(\mathbf{A},\phi)$ and introduce small deviations of all the quantities with respect to the initial state $n=n_0$, $\mathbf{u}=0$, $\mathbf{A}=\mathbf{A}_0(\mathbf{r}) = (\mathbf{B}_0\times\mathbf{r})/2$, $\phi=0$; then we write $\mathbf{u}=\hat{\mathbf{e}}_z u_{1z}+\mathbf{u}_{1\perp}$, $\mathbf{A}=\mathbf{A}_0+\hat{\mathbf{e}}_z A_{1z}+\mathbf{A}_{1\perp}$, $\phi=\phi_1$. Furthermore, we assume the long beam limit, i.e. $\partial^2/\partial\xi^2 \ll \omega_p^2/c^2 \equiv k_p^2$ (ω_p being the electron plasma frequency), which under suitable boundary conditions implies that $|\mathbf{A}_{1\perp}| \ll A_{1z}$ and $|\mathbf{u}_{1\perp}| \ll u_{1z}$. Consequently, after linearizing the Lorentz-Maxwell system of equations, we obtain the equation for the dimensionless *wake potential* U_w, driven by the beam density, i.e., $\left(\nabla_\perp^2 - \kappa^2\right)U_w = \kappa^2\rho_b/n_0\gamma_0$, where ω_{UH} is the upper hybrid frequency, $\kappa \equiv \omega_p^2/\omega_{UH}c^2$, $\nabla_\perp$ is the transverse part of the gradient operator, and $U_w = (A_{1z}-\phi_1)/m_0\gamma_0 c^2$ with $A_{1z}=A_{1z}(\mathbf{r}_\perp,\xi)$ and $\phi_1=\phi_1(\mathbf{r}_\perp,\xi)$ the perturbations of both the longitudinal component of the vector potential and the electric potential, respectively. Here, $\mathbf{r}_\perp$ is the transverse position vector, $\xi = z - \beta ct \simeq z - ct$ plays the role of time-like variable, m_0 and γ_0 are the rest mass and the unperturbed relativistic gamma factor of the single particle in the electron/positron beam, respectively. To provide a self consistent description of the transverse electron/positron beam dynamics, we have to consider the total force acting on the single particle of the beam, that accounts for the transverse gradient of U_w as well as the effects produced by the external magnetic field $\mathbf{B}_0$. To this end, we describe the transverse beam dynamics by means of the Thermal Wave Model (TWM) [2, 3].

TWM provides an effective description of the charged-particle beam dynamics in terms of a complex function, say Ψ, called beam wave function (BWF). The transverse spatio-temporal evolution of the BWF is given by $i\varepsilon\partial\Psi/\partial\xi = \mathscr{H}(\mathbf{r}_\perp, -i\varepsilon\nabla_\perp, \xi)\Psi$, where, for normalized Ψ, $\rho_b(\mathbf{r}_\perp,\xi) = (N/\sigma_z)|\Psi(\mathbf{r}_\perp,\xi)|^2$ (N and σ_z being the total number of particles and the beam length, respectively) [2, 3], $\mathscr{H}(\mathbf{r}_\perp, \mathbf{p}_\perp, \xi)$ is the effective Hamiltonian describing the perturbed transverse motion of a single particle of the beam, and ε the transverse beam emittance. Provided that the longitudinal dynamics is ignored, i.e. $\mathbf{p} = \hat{\mathbf{e}}_z m_0\gamma_0 c + \mathbf{p}_{1\perp}$, it is easily seen that, $\mathscr{H} = \Delta H/H_0 = (H - H_0)/H_0$ is the relative first-order perturbation of the single electron/positron Hamiltonian $H(\mathbf{r},\mathbf{p},\xi) = c\sqrt{(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + m_0^2c^2} + q\phi$, where $H_0 = m_0\gamma_0c^2$ is the initial unperturbed total energy of the single particle of the beam. Consequently, the above equations for BWF and U_w can be cast as the following *Zakharov system* of equations, viz.,

$$i\varepsilon\frac{\partial\psi_m}{\partial\xi} = -\frac{\varepsilon^2}{2}\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\left(r_\perp\frac{\partial\psi_m}{\partial r_\perp}\right) + U_w\psi_m + \left(\frac{1}{2}Kr_\perp^2 + \frac{m^2\varepsilon^2}{2r_\perp^2}\right)\psi_m. \quad (1)$$

$$\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\left(r_\perp\frac{\partial U_w}{\partial r_\perp}\right) - \kappa^2U_w = \kappa^2\frac{N}{n_0\gamma_0\sigma_z}|\psi_m|^2, \quad (2)$$

where we have put $\Psi(r_\perp,\varphi,\xi) = \exp\left[im(\varphi - k_c\xi/2)\right]\psi_m(r_\perp,\xi)$ with m integer and $K = (\omega_c/2\gamma_0c)^2 \equiv \left(qB_0/2m_0\gamma_0c^2\right)^2 \equiv (k_c/2)^2$. This system governs the self consistent *spatio temporal* evolution of the *PWF self-interaction* of the electron/positron beam. In principle, once eq.(2) is solved for U_w, we get the functional $U = U\left[|\psi_m|^2\right]$ that makes eq. (1) the generalized nonlinear Schrödinger equation (NLSE). Note that, due to the *helicity* phase factor $\exp\left[im(\varphi - k_c\xi/2)\right]$ for $m \neq 0$ the BWF describes vortices states associated with the orbital angular momentum of the beam particles in the external magnetic field (m plays the role of *vortex charge*). Based on the pair of eqs. (1) and (2), an investigation, both analytical and numerical, has been carried out, taking into account the diverse limiting cases. Hereafter, and in Ref. [4], we give a summary of the maim results.

First of all, in the linear limit, i.e. $U = U\left[|\psi_m|^2\right] \simeq 0$ (the beam is travelling along $\mathbf{B}_0$ and the self interaction is assumed negligible), for arbitrary integer m, the pair of eqs. (1) and (2) reduces to a linear Schrödinger equation that admits solutions in the form of Laguerre-Gauss modes, say $\psi_{p,m}(r_\perp,\xi)$ (p is an arbitrary integer), whose transverse size, $\sigma_{p,m}(\xi)$, is proportional to the one associated with the purely Gaussian mode ($m = p = 0$), say $\sigma(\xi)$. The latter satisfies the following envelope equation: $d^2\sigma/d\xi^2 + K\sigma - \varepsilon^2/\sigma^3 = 0$. For a long enough region along z, it describes the typical *sausage-like* transverse beam size modulations (betatron oscillations), preserving the collapse (focusing to a single point). Fig. 1 shows the density plot for $|\psi_{p,m}|^2$ in the normalized plane $(x/\sigma,\, y/\sigma)$ for some combinations of p and m. For a given m, the number of rings increases as p increases. When $m = 0$, the structure of the density plots is always constituted by a central core plus p rings, whilst for $m \neq 0$, the density plots are associated with the diverse states of vortices.

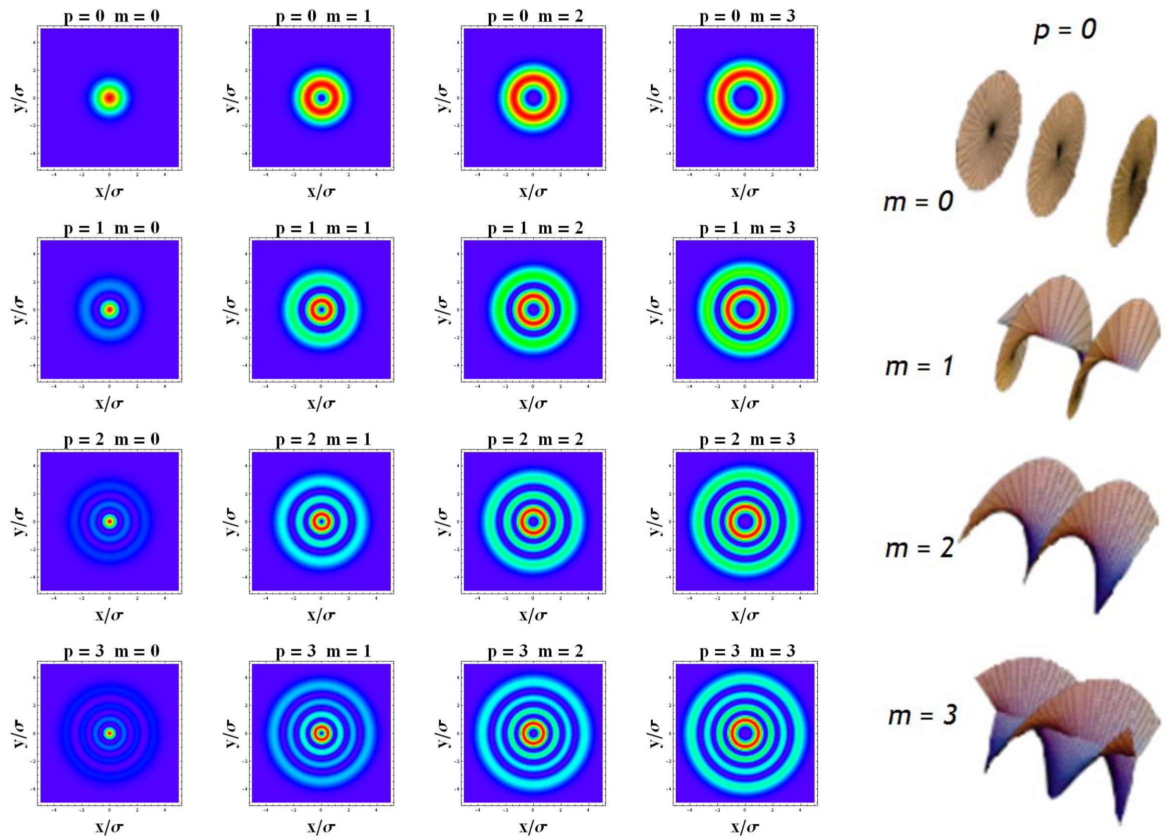

Figure 1: *Left: Density plots for $|\Psi_{m,p}|^2$ in the normalized plane $(x/\sigma,\, y/\sigma)$ for some combinations of p and m. Right: qualitative representation of the BWF phase for different values of the vortex charge m when* $p = 0$.

When the self interaction is not negligible, one can suitably investigate two limiting cases.
(i). If the transverse beam size is much greater than the plasma wavelength, i.e. $k_p\sigma \gg 1$, the system (1) and (2) reduces to the 2D *Gross-Pitaevskii equation*, i.e.,

$$i\varepsilon\frac{\partial\psi_m}{\partial\xi} = -\frac{\varepsilon^2}{2}\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}\left(r_\perp\frac{\partial\psi_m}{\partial r_\perp}\right) - \frac{N}{\sigma_z n_0\gamma_0}\,|\,\psi_m\,|^2\,\psi_m + \left(\frac{1}{2}Kr_\perp^2 + \frac{m^2\varepsilon^2}{2r_\perp^2}\right)\psi_m\,.$$

In Ref.[4] we carry out a numerical analysis of this sort of NLSE, showing the existence of vortices, nonlinear coherent states (2D solitons) and beam halos. here, from the virial equation associated to this NLSE, we find the following envelope equation:

$$d^2\sigma_m^2/d\xi^2 + 4K\sigma_m^2 = 4\mathscr{A}_m\,,$$

where $\mathscr{A}_m \equiv \pi\varepsilon^2 \int_0^\infty \left[|\partial\psi_m/\partial r_\perp|^2 + (m^2/r_\perp^2)|\psi_m|^2\right] r_\perp dr_\perp - (\pi N/\sigma_z n_0\gamma_0)\int_0^\infty |\psi_m|^4 r_\perp dr_\perp + (K/2)\sigma_m^2$ is constant of motion, i.e. $d\mathscr{A}_m/d\xi = 0$, and $\sigma_m \equiv \left[2\pi\int_0^\infty r_\perp^2 |\psi_m(r_\perp,\xi)|^2 r_\perp dr_\perp\right]^{1/2}$ is the transverse beam size. Note that the interplay between the positive and the negative terms in the definition of $\mathscr{A}_m$ can make this quantity positive, negative or zero. If $\mathscr{A}_m > 0$, in correspondence of the initial conditions $\sigma_0 \equiv \sigma_m(0)$, and $\sigma_0' \equiv (d\sigma_m/d\xi)_{\xi=0}$, the envelope equation describes stable oscillations in σ_m with frequency $2\sqrt{K}$, in the range $0 < \frac{1}{2}\sigma_0'^2 + \frac{1}{2}K\sigma_0^2 < \mathscr{A}_m$, whilst σ_m would reach zero in a finite time in the range $0 < \mathscr{A}_m < \frac{1}{2}\sigma_0'^2 + \frac{1}{2}K\sigma_0^2$. If $\mathscr{A}_m < 0$, always σ_m would reach zero in a finite time. The latter cases would correspond to a collapse instability. However, as σ_m reaches very small values, the condition $k_p\sigma_m \gg 1$ is no longer satisfied and the collapse does not take place. It is interesting to describe the self-interaction of an electron/positron beam that enters a thin plasma slab (*plasma lens*) of length l at $\xi = 0$ where, as initial conditions, is assumed that $\sigma_0' = 0$ and the BWF is purely Gaussian, $\psi(r_\perp,0) = \exp[-r_\perp^2/2\sigma_0^2]/\sqrt{\pi}\sigma_0$. Provided that$\sqrt{K}l \ll 1$, the self-interaction is very short, namely the the BWF remains almost unchanged except for the appearance of a *chirping phase factor*, that accounts for a strong change in the particle momentum distributions in the transverse plane (*kick approximation*), viz.,$\psi(r_\perp,l) \simeq \psi(r_\perp,0)\exp[ir_\perp^2/2\varepsilon\rho(l) + i\phi(l)]$, where $\rho(\xi)$ and $\phi(\xi)$ are the curvature radius of the wavefront and a homogeneous phase at location ξ, respectively, such that, as $\xi \to 0$, $\rho \to \infty$ and $\phi \to 0$. At the lens exit ($\xi = l$), the transverse beam size is $\sigma(l) \simeq \sqrt{\sigma_0^2 + 2\left(\mathscr{A}_0 - K\sigma_0^2\right)l^2}$ ($\mathscr{A}_0$ being the constant of motion corresponding to the Gaussian beam). Consequently, under the condition $\mathscr{A}_0 - K\sigma_0^2 < 0$, the beam is focussed and out of the lens it reaches a minimum spot size that is greater than zero, according to the envelope equation in the vacuum (i.e., no plasma and $K = 0$), viz., $d^2\sigma/d\xi^2 - \varepsilon^2/\sigma^3 = 0$. We have *weak focusing* (*strong focusing*) if the above condition is satisfied for positive (negative) values of $\mathscr{A}_0$.

(**ii**). If $k_p\sigma \ll 1$, eqs. (1) and (2) reduces to a nonlocal NLS equation whose aberrationless (i.e. Gaussian) approximate solution leads to the envelope equation $d^2\sigma/d\xi^2 + K\sigma + \eta/\sigma - \varepsilon^2/\sigma^3 = 0$, where $\eta = \left(2e^2\omega_p^2 N/m_0\gamma_0\omega_{UH}^2\sigma_z\right)$. It admits the *self-equilibrium solution* $\sigma_{eq}' = \sqrt{-\mu + \sqrt{1+\mu^2}}\,\sigma_{eq}$, where $\sigma_{eq} \equiv \varepsilon^{1/2}/K^{1/4}$ is the corresponding self-equilibrium solution for the linear case $\eta = 0$ and $\mu = \eta/2\varepsilon\sqrt{K}$. It is easily seen that $\sigma_{eq}' < \sigma_{eq}$ in all the range $0 \leq \mu < \infty$. This implies that the term proportional to $1/\sigma$ accounts for the squeezing of the beam.

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