Qualitative geometry of qa, qh and qi configurations

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Abstract

A qualitative invariant description of the characteristic geometries of quasi-axi- symmetric, quasi-helically symmetric and quasi-isodynamic stellarator configurations is given in terms of the first order of the distance from the magnetic axis. Relations constraining the following four functions are obtained: the curvature and the torsion of the magnetic axis, the ellipticity and the orientation of the flux-surface cross-section as functions of arc length along the magnetic axis.

Introduction

Mercier's expansion around the magnetic axis of a 3d magnetic confinement configuration [1] employs the metric

$$\mathrm{d}\vec{r}^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\vartheta^2 + 2\tau\rho^2 \mathrm{d}l\mathrm{d}\vartheta + [(1-\kappa\rho\cos\vartheta)^2 + \tau^2\rho^2]\mathrm{d}l^2$$

(κ and τ curvature and torsion (positive for right-handed helicity) of this axis, ρ and ϑ polar coordinates in a plane perdendicular to the axis and l the arclength along this axis) and yields $B\kappa\rho\cos\vartheta$ (with B the field strength along the axis) for the field strength in first order in ρ .

Transformation [2] of this quantity to Hamada or (with nearly identical algebra) magnetic coordinates yields

$$\phi = \frac{1}{I}\int B\mathrm{d}l \quad \text{and} \quad B\kappa\rho\cos\vartheta = B\kappa\xi(l,\theta)s^{\frac{1}{2}}$$

in magnetic coordinates with I the poloidal current encompassed by the magnetic axis, ϕ the toroidal magnetic coordinate, s the normalized toroidal flux F_t , θ the poloidal magnetic coordinate and

$$\xi = \xi_c \cos 2\pi\theta + \xi_s \sin 2\pi\theta$$

with

$$\xi_c = \sqrt{F_\mathrm{t}'/\pi B}(\sqrt{e}\sin K \sin\alpha + \cos K \cos\alpha/\sqrt{e})$$

$$\xi_s = \sqrt{F_\mathrm{t}'/\pi B}(\sqrt{e}\cos K \sin\alpha - \sin K \cos\alpha/\sqrt{e})$$

where e is the half-axis ratio of the elliptical (to lowest order) flux surface cross-section (with the convention $e > 1$), α the angle counted from the binormal of the magnetic axis to the major half-axis and K is related to the rotational transform ι (counted positive for left-handed helicity of the configuration) through

$$dK/dl = (dJ/dF_\mathrm{t} - 2\tau + 2d\alpha/dl)/(e+1/e) + 2\pi B\iota/I$$

with J the toroidal current through the flux surface cross-section and K satisfying the boundary condition

$$K(L)-K(0)=2\pi m+\alpha(L)-\alpha(0)$$

where $\alpha(L)-\alpha(0)=n\pi$ ($n/2$ number of full turns of elliptical cross-section, m number of turns of the normal to the magnetic axis, L the length of the magnetic axis).

So, to this order, the qa, qh and qi conditions constrain the quantities $B(l)$ and $B\kappa\xi$.

Results

i) qa

In qa symmetry B is not allowed to be a function of ϕ so that it is constant and $\kappa\xi$ is a function of θ only, which results in $\kappa\xi_c$ and $\kappa\xi_s$ being constants. Therefore,

$$\kappa = const/\sqrt{e\ \sin^2\alpha+(1/e)\ \cos^2\alpha}$$

so that the curvature is a nonvanishing function of the elliptic shape and its turning angle. For stellarator symmetric configurations with the convention $\alpha=K=0$ for $l=0$,

$$\sin K=\sqrt{e}\ \sin\alpha/\sqrt{e\ \sin^2\alpha+(1/e)\ \cos^2\alpha}$$

So, since $m=0$ for this quasi-symmetry, the magnetic axis is a perturbed toroidal circle and the above constraint determines the torsion for vanishing longitudinal current density on the magnetic axis. For constant ellipticity and turning rate ($n=1$ per period) (which characterizes the simplest so called $\mathrm{l}=2$ stellarator) $K=\alpha$ in contradiction to the above constraint so that the magnetic axis must exhibit torsion for achieving qa symmetry.

Figure 1 shows an example of a qa equibrium [3]; the four functions κ, τ, e and α and their two above constraints are plotted. Since the torsion changes sign the functions α and K exhibit quite similar behabior.

ii) qh

Here, again, B is not allowed to be a function of ϕ so that it is constant and $\phi=l/L$. $\kappa\xi$ is a function of $\theta+\phi$ only, which, for stellator symmetry, results in $\kappa\xi_c=const\cos 2\pi\phi$ and $\kappa\xi_s=-const\sin 2\pi\phi$. Therefore the constraint for the curvature is the same as in qa so that, again the curvature is a nonvanishing function [4,5]. Since $m=1$ (per period) for this quasi-symmetry, the magnetic axis is a perturbed helix. The constraint for K becomes

$$\sin K=[\sqrt{e}\ \sin\alpha\ \cos(2\pi l/L)+\sqrt{1/e}\ \cos\alpha\ \sin(2\pi l/L)]/\sqrt{e\ \sin^2\alpha+(1/e)\ \cos^2\alpha}$$

Figure 2 shows an example of a qh equibrium [6]. Since the torsion does not change sign the secular behavior of K and α ($n=-1$)is opposite.

iii) qi

Quasi-isodynamicity - in the order considered here - requires stationarity of the second adiabatic invariants of all reflected particles in the neighborhood of the magnetic axis. In case of stellarator symmetry, this condition is satisfied if the even (with respect to l) part of $B\kappa\xi$

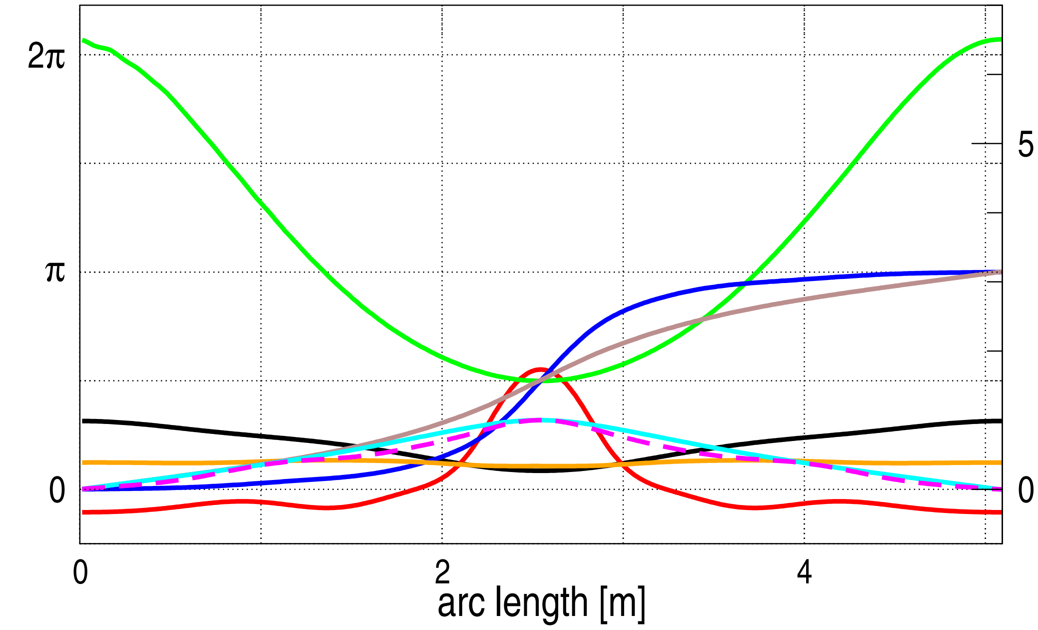

Figure 1: Curvature (black), torsion (red), half-axis ratio (green), angle α (dark blue), the function K (brown), $\kappa[e\sin^2\alpha + (1/e)\cos^2\alpha]^{\frac{1}{2}}$ (orange), $\sin K$ (cyan) and its constraint (magenta, broken) as functions of arclength along one period.

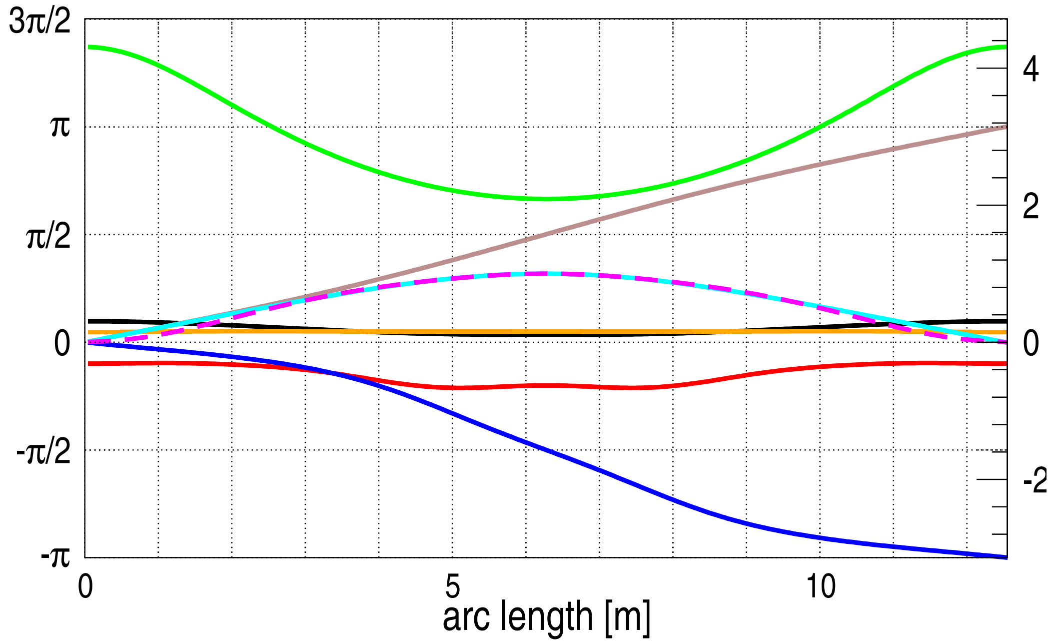

Figure 2: See Figure 1.

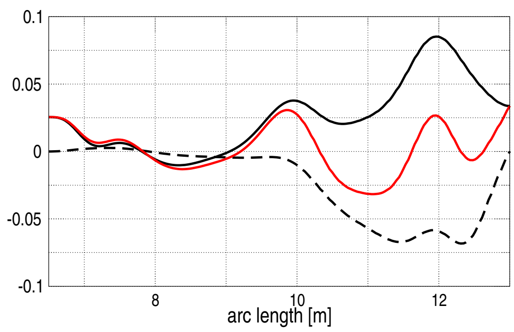

Figure 3: The two parts of the qi equation (without the positive definite factor $\sqrt{B}$), first line (black), second line (broken black) and their sum (red) between the minimum of B (left) and the maximum of B (right), ie. along half a period.

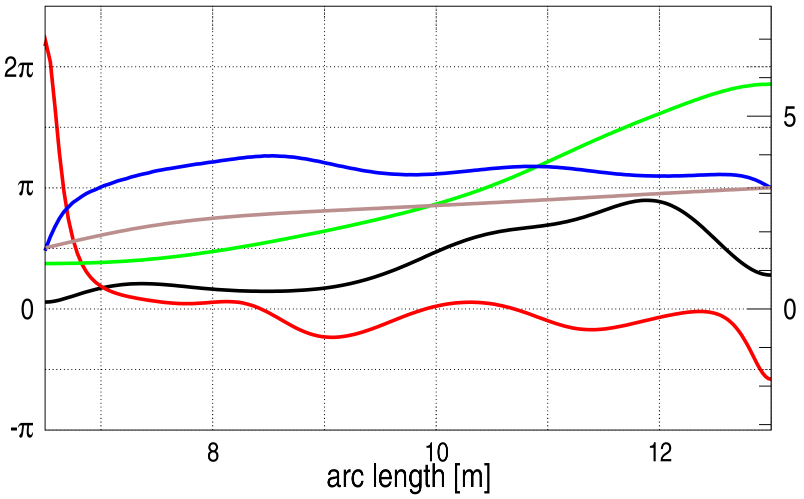

Figure 4: See Figure 1; instead of κ $10\cdot\kappa$ *is plotted.*

evaluated along field lines vanishes:

$$\sqrt{B}\kappa[\sqrt{e}\ \sin K\ \sin\alpha + \cos K\ \cos\alpha/\sqrt{e}]\cos(2\pi\iota\int Bdl/\oint Bdl)$$

$$-\sqrt{B}\kappa[\sqrt{e}\ \cos K\ \sin\alpha - \sin K\ \cos\alpha/\sqrt{e}]\sin(2\pi\iota\int Bdl/\oint Bdl) = 0$$

Figure 3 shows the evaluation of the qi configuration obtained by integrated optimization of several physics goals described in [7]. The tendency for cancellation of the two terms of the qi equation is clearly seen but far from perfect as has to be expected since the optimization was not excusively dedicated to achieving quasi-isodynamicity. Figure 4 shows the constituents of the qi equation and tendencies to be expected: i) the curvature cannot be concentrated near the extrema of B because of the confinement requirement for deeply trapped particles and requirement of elimination of transiting particles; ii) the curvature must be concentrated at significant ellipticity of the cross-sections because of the requirement of MHD stability; iii) the favorable influence of K and α being near π for significant ellipticity is obvious.

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