Neoclassical theory of plasma rotation due to internal and external sources

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A method for calculating the plasma rotation velocity caused by the effect of fluctuations due to instabilities and/or by externally imposed sources is presented for multiple ion species plasmas in a general toroidal magnetic field. We assume that the distribution function is perturbed from the Maxwell distribution function f_{a0} by the presence of sources. Then we start with the following drift kinetic equation for a perturbed distribution function f_{a1} for species a:

$$\mathcal{V}(f_{a1}) - \sum_b C_{ab}(f_{a1}, f_{b1}) = S_a + L_a$$

with $\mathcal{V} = \left[v_\parallel \boldsymbol{b} + c\Phi'(\rho)\boldsymbol{B} \times \nabla\rho/\langle B^2\rangle\right] \cdot \nabla - v(1-\xi^2)/(2B^2)\,\boldsymbol{B}\cdot\nabla B\,\partial/\partial\xi$, where $\boldsymbol{b} = \boldsymbol{B}/B$; $v_\parallel = \boldsymbol{b}\cdot\boldsymbol{v}$; $\xi = v_\parallel/v$; $\Phi(\rho)$ is the electrostatic potential; the constant ρ labels a magnetic surface; $\langle\cdot\rangle$ denotes the flux-surface average; and S_a and L_a represent the source term driving plasma rotation and the loss term. In this formulation we use an approximate momentum-conserving form [1] for the linearized Fokker-Planck collision operator $C_{ab}(f_{a1}, f_{b1}) \equiv C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1})$:

$$C_{ab}(f_{a1}, f_{b1}) \simeq \nu^D_{ab}(v)\mathcal{L}(f_{a1}) + \xi\left[C^1_{ab}(f^1_{a1}, f^1_{b1}) + \nu^D_{ab}(v) f^1_{a1}\right] + P_2(\xi)\,\Delta C_{ab}(f^2_{a1}, f^2_{b1}),$$

where $C_{ab}(P_l(\xi)\varphi_a(v), P_l(\xi)\varphi_b(v)) = P_l(\xi)C^l_{ab}(\varphi_a(v), \varphi_b(v))$; $P_l(\xi)$ is the Legendre polynomials; $f^l_{a1} = (l+1/2)\int_{-1}^{1} P_l(\xi) f_{a1} d\xi$; $\Delta C_{ab}(f^2_{a1}, f^2_{b1}) = C^2_{ab}(f^2_{a1}, f^2_{b1}) + 3\nu^D_{ab}(v) f^2_{a1}$; $\mathcal{L}$ is the pitch-angle scattering operator; and $\nu^D_{ab}(v)$ is the deflection collision frequency. We further approximate the collision operator $C^2_{ab}(f^2_{a1}, f^2_{b1})$ by the energy-dependent Krook term [2] $-\nu^T_{ab}(v) f^2_{a1}$. The collision term $P_2(\xi)\Delta C_{ab}$ can be neglected in the low collisionality regime although it should be retained in th collisional (Pfirsch-Schlüter) regime. We model the loss term by an external drag with characteristic frequency ν_{a0} in the form: $L_a = -\nu_{a0}\,\xi\, f^1_{a1}$. It is noted here that the suffixes with the Roman indices a and b represent the electron and ion species, and those with the Greek indices α, β and γ represent only ion species.

Let us introduce the following auxiliary equations:

$$\mathcal{V}(f^{(k)}_{sa}) + \sum_b C_{ab}(f^{(k)}_{sa}, f^{(k)}_{sb}) - \nu_{a0}\,\xi\, f^{(k)1}_{sa} = \frac{e_a}{T_a} B v_\parallel \left(\frac{v^2}{v_a^2} - \frac{5}{2}\right)^k f_{a0} \qquad (k = 0, 1) \qquad (1)$$

and

$$\mathcal{V}(h_e) + C_e(h_e) - P_2(\xi)\tilde{\nu}_e^T(v)h_e^2 - \nu_{e0}\,\xi\,h_e^1 \;=\; -\xi\sum_\alpha{}' \left[C^1_{e\alpha}(f_{se}^{(0)1}, f_{s\alpha}^{(0)1}) + \nu^D_{e\alpha}(v) f_{se}^{(0)1}\right], \quad (2)$$

where $C_e(h_e) = \xi[C^1_{ee}(h^1_e, h^1_e) + \nu^D_{ee}(v)h^1_e] + \nu^D_e(v)\,\mathcal{L}\,(h_e)$; $\tilde{\nu}^T_{ea}(v) = \nu^T_{ea}(v) - 3\nu^D_{ea}(v)$; $\nu^D_a(v) = \sum_b \nu^D_{ab}(v)$; $\tilde{\nu}^T_a(v) = \sum_b \tilde{\nu}^T_{ab}(v)$; $\sum'_\alpha$ means the summation only over ions; and $e_a = Z_\alpha e$, m_a, T_a and $v_a = \sqrt{2T_a/m_a}$ are the charge, the mass , the temperature and the thermal velocity for species a.

The solutions to the equations (1) and (2) can be written in the form [3]:

$$f_{sa}^{(k)} \;=\; -\frac{1}{\langle B^2\rangle}\frac{1}{f^\star_{ca}}\frac{\nu^D_a(v)}{\nu_{aa}}\;K_{sa}^{(k)}(v)\;g_a, \quad (3)$$

$$h_e \;=\; -\frac{1}{\langle B^2\rangle}\frac{1}{f^\star_{ce}}\frac{\nu^D_e(v)}{\nu_{ee}}\;K_h(v)\;g_e \quad (4)$$

with

$$f^\star_{ca} \;=\; -\frac{\nu^D_a(v)}{\nu_{aa}}\frac{1}{\langle B^2\rangle}\frac{3}{2}\,\langle\, B\int_{-1}^{1} d\xi\,\xi\,g_a\,\rangle,$$

where $K_{sa}^{(k)}(v) = \langle\, B f_{sa}^{(k)1}\,\rangle$, $K_h(v) = \langle\, B h^1_e\,\rangle$, $\nu_{ab} = 4\pi n_b e_a^2 e_b^2 \ln\Lambda/(m_a^2 v_a^3)$; n_b is the number density, and g_a is the solution to the kinetic equation with *the pitch-angle-scattering and Krook collision terms*:

$$\mathcal{V}(g_a) + \nu^D_a(v)\,\mathcal{L}\,(g_a) - P_2(\xi)\tilde{\nu}^T_a(v)g_a^2 \;=\; \nu_{aa}\,B\,\xi. \quad (5)$$

Using the relations $f_{sa}^{(k)1} = B\,K_{sa}^{(k)}/\langle B^2\rangle$ and $h^1_e = B\,K_h/\langle B^2\rangle$, we find the functions $K_{sa}^{(k)}(v)$ and $K_h(v)$ to satisfy the equations

$$\frac{f^\star_{ta}}{f^\star_{ca}}\,\nu^D_a(v)\,K_{sa}^{(k)} - \sum_b\; C^1_{ab}(\,K_{sa}^{(k)},\,K_{sb}^{(k)}\,) + \nu_{a0}K_{sa}^{(k)} = -\frac{e_a}{T_a}\,\langle B^2\rangle\,v\left(\frac{v^2}{v_a^2} - \frac{5}{2}\right)^k f_{a0}, \quad (6)$$

$$\frac{f^\star_{te}}{f^\star_{ce}}\,\nu^D_e(v)\,K_h - C^1_e\,(\,K_h) + \nu_{e0}K_h \;=\; \sum_\alpha{}'\;\left[\,C^1_{e\alpha}(\,K_{se}^{(0)},\,K_{s\alpha}^{(0)}\,) + \nu^D_{e\alpha}(v)\,K_{se}^{(0)}\,\right], \quad (7)$$

where $f^\star_{ta} = 1 - f^\star_{ca}$.

Noting the relation

$$\langle \int d\boldsymbol{v}\frac{h_e}{f_{e0}}S_e\,\rangle \simeq \frac{m_e}{n_e T_e}\frac{1}{\langle B^2\rangle}\frac{1}{\tau_{ee}}\left[\langle B n_e u_{\parallel e}\rangle - \frac{3}{5}\frac{\langle B q_{\parallel e}\rangle}{T_e}\right]\sum_\alpha{}' Z_\alpha^2\langle B n_\alpha u_{\parallel\alpha}^{(0)}\rangle,$$

we can obtain the simultaneous linear equations for the parallel ion flows $\langle B n_\alpha u_{\parallel\alpha}\rangle = \langle B\int d\boldsymbol{v} v_\parallel f_{\alpha 1}\rangle$ and $\langle B q_{\parallel\alpha}\rangle = \langle B\int d\boldsymbol{v} v_\parallel (m_\alpha v^2/2 - 5T_\alpha/2) f_{\alpha 1}\rangle$:

$$\sum_\beta{}' \begin{bmatrix} L^{\alpha\beta}_{11} & L^{\alpha\beta}_{12} \\ L^{\alpha\beta}_{21} & L^{\alpha\beta}_{22}\end{bmatrix}\begin{bmatrix}\langle B n_\beta u_{\parallel\beta}\rangle \\ \frac{2}{5}\frac{\langle B q_{\parallel\beta}\rangle}{T_\beta}\end{bmatrix} = \frac{n_\alpha T_\alpha}{m_\alpha}\tau_{\alpha\alpha}\langle B^2\rangle \left\{ \left({}^t U_\alpha\right)^{-1}\;\langle \int d\boldsymbol{v}\,\frac{S_\alpha}{f_{\alpha 0}}\begin{bmatrix} f_{s\alpha}^{(0)} \\ f_{s\alpha}^{(1)}\end{bmatrix}\rangle\right.$$

$$+ \frac{T_e}{T_\alpha} \frac{1}{\sum'_\gamma (e_\gamma^2/e_\alpha^2)\langle B n_\gamma u_{\parallel\gamma}^{(0)}\rangle} \langle \int d\boldsymbol{v} \frac{h_e}{f_{e0}} S_e \rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Bigg\}, \tag{8}$$

where $\tau_{ab}^{-1} = (4/3\sqrt{\pi})\nu_{ab}$; $\langle B n_\alpha u_{\parallel\alpha}^{(k)}\rangle = \langle B \int d\boldsymbol{v} v_\parallel f_{s\alpha}^{(k)}\rangle$, $\langle B q_{\parallel\alpha}^{(k)}\rangle = \langle B \int d\boldsymbol{v} v_\parallel (m_\alpha v^2/2 - 5T_\alpha/2) f_{s\alpha}^{(k)}\rangle$, and

$$U_\alpha = \begin{bmatrix} \langle B n_\alpha u_{\parallel\alpha}^{(0)}\rangle & \langle B n_\alpha u_{\parallel\alpha}^{(1)}\rangle \\ \frac{2}{5}\frac{\langle B q_{\parallel\alpha}^{(0)}\rangle}{T_\alpha} & \frac{2}{5}\frac{\langle B q_{\parallel\alpha}^{(1)}\rangle}{T_\alpha} \end{bmatrix}.$$

The coefficients $L_{ij}^{\alpha\beta}$ in (8) are defined by

$$\begin{aligned} L_{11}^{\alpha\beta} &= \left(\mu_{\alpha1}^\star - M_\alpha^{00} + \nu_{\alpha0}\,\tau_{\alpha\alpha}\right)\delta_{\alpha\beta} - \frac{e_\beta^2}{e_\alpha^2} N_{\alpha\beta}^{00}, \\ L_{12}^{\alpha\beta} &= \left(\mu_{\alpha2}^\star + M_\alpha^{01}\right)\delta_{\alpha\beta} + \frac{e_\beta^2}{e_\alpha^2} N_{\alpha\beta}^{01}, \\ L_{21}^{\alpha\beta} &= \left(\mu_{\alpha2}^\star + M_\alpha^{10}\right)\delta_{\alpha\beta} + \frac{e_\beta^2}{e_\alpha^2} N_{\alpha\beta}^{10}, \\ L_{22}^{\alpha\beta} &= \left(\mu_{\alpha3}^\star - M_\alpha^{11} + \frac{5}{2}\nu_{\alpha0}\,\tau_{\alpha\alpha}\right)\delta_{\alpha\beta} - \frac{e_\beta^2}{e_\alpha^2} N_{\alpha\beta}^{11}, \end{aligned}$$

where $M_\alpha^{ij} = \sum_b (\tau_{\alpha\alpha}/\tau_{\alpha b}) M_{\alpha b}^{ij}$,

$$\begin{bmatrix} \mu_{a1}^\star \\ \mu_{a2}^\star \\ \mu_{a3}^\star \end{bmatrix} = \frac{\tau_{aa}}{n_a}\frac{8\pi}{3}\int_0^\infty \frac{f_{ta}^\star}{f_{ca}^\star}\,\nu_a^D(v)\,\frac{v^4}{v_a^2} \begin{bmatrix} 1 \\ v^2/v_a^2 - 5/2 \\ (v^2/v_a^2 - 5/2)^2 \end{bmatrix} f_{a0}\,dv,$$

and the matrix elements M_{ab}^{ij} and N_{ab}^{ij} are given by [2]

$$\begin{aligned} \frac{n_a}{\tau_{ab}} M_{ab}^{ij} &= \int v_\parallel L_i^{3/2}\left(\frac{v^2}{v_a^2}\right) C_{ab}\left[\frac{2v_\parallel}{v_a^2} L_j^{3/2}\left(\frac{v^2}{v_a^2}\right) f_{a0}, f_{b0}\right] d\boldsymbol{v}, \\ \frac{n_a}{\tau_{ab}} N_{ab}^{ij} &= \int v_\parallel L_i^{3/2}\left(\frac{v^2}{v_a^2}\right) C_{ab}\left[f_{a0}, \frac{2v_\parallel}{v_b^2} L_j^{3/2}\left(\frac{v^2}{v_b^2}\right) f_{b0}\right] d\boldsymbol{v} \end{aligned}$$

with the associate Laguerre polynomials $L_j^{3/2}(v^2/v_a^2)$ of order 3/2.

Equations (6) and (7) can be solved by expanding the functions $K_{sa}^{(k)}(v)$ and $K_h(v)$ in the series of the associated Laguerre polynomials of order 3/2. Truncation after only first two terms in the expansion yields sufficient accuracy for the calculation of the parallel flows $\langle B n_a u_{\parallel a}^{(k)}\rangle$ and $\langle B q_{\parallel a}^{(k)}\rangle$. Then the ion flows $\langle B n_\alpha u_{\parallel\alpha}^{(k)}\rangle$ and $\langle B q_{\parallel\alpha}^{(k)}\rangle$ are obtained by solving the following simultaneous linear equations

$$\sum_\beta{}' \begin{bmatrix} L_{11}^{\alpha\beta} & L_{12}^{\alpha\beta} \\ L_{21}^{\alpha\beta} & L_{22}^{\alpha\beta} \end{bmatrix} U_\beta = -\frac{e_\alpha n_\alpha}{m_\alpha}\langle B^2\rangle\,\tau_{\alpha\alpha} \begin{bmatrix} c_1 & c_2 \\ 0 & \frac{2}{5} \end{bmatrix},$$

where

$$c_1 = 1 - \frac{Z_\alpha}{D_e}\left(\mu^\star_{e3} + \frac{3}{2}\mu^\star_{e2} + \sqrt{2} + \bar{Z} + \frac{5}{2}\nu_{e0}\,\tau_{ee}\right)$$

and $c_2 = 5(\mu^\star_{e2} + 3\mu^\star_{e1}/2 + 3\nu_{e0}\tau_{ee}/2)/2$ with $D_e = (\mu^\star_{e1} + \bar{Z} + \nu_{e0}\,\tau_{ee})(\mu^\star_{e3} + \sqrt{2} + 13\bar{Z}/4 + 5\,\nu_{e0}\,\tau_{ee}/2) - (\mu^\star_{e2} - 3\bar{Z}/2)^2$ and the effective charge $\bar{Z} = \sum_\alpha{}' n_\alpha Z_\alpha^2/n_e$.

The first-two-terms approximation in the series expansions for the functions $K_{sa}^{(k)}(v)$ and $K_h(v)$ is appropriate for calculating the low velocity moments of these functions. However, more accurate functions $K_{sa}^{(k)}(v)$ and $K_h(v)$ are generally required for calculating the integrals involving the source terms S_a in (8). For this purpose we use the functions $K_{sa}^{(k)}(v)$ and $K_h(v)$ obtained by taking sufficient large expansion terms. In addition, we must evaluate the functions g_a for these integrals including sources. The equation (5) for g_a can be solved analytically for restricted asymptotic regimes. For general toroidal plasmas this equation is solved using numerical codes, for example, the Drift Kinetic Equation Solver (DKES).

The parallel ion flow for a single ion species plasma has the simple form. Multiplying Eq.(8) by ${}^tU_a$ and noticing the relation

$${}^tU_\alpha \begin{bmatrix} L_{11}^{\alpha\alpha} & L_{12}^{\alpha\alpha} \\ L_{21}^{\alpha\alpha} & L_{22}^{\alpha\alpha} \end{bmatrix} = -\frac{e_\alpha n_\alpha}{m_\alpha}\langle B^2\rangle\,\tau_{\alpha\alpha}\begin{bmatrix} c_1 & 0 \\ c_2 & \frac{2}{5} \end{bmatrix},$$

we find

$$\langle B n_\alpha u_{\parallel\alpha}\rangle = -\frac{1}{c_1}\frac{T_\alpha}{e_\alpha}\left\{\langle\int d\boldsymbol{v}\,S_\alpha\frac{f_{s\alpha}^{(0)}}{f_{\alpha 0}}\rangle + \frac{T_e}{T_\alpha}\langle\int d\boldsymbol{v}\,S_e\frac{h_e}{f_{e0}}\rangle\right\},$$

where we have used $L_{12}^{\alpha\alpha} = L_{21}^{\alpha\alpha}$. This expression is equivalent to that obtained in [4].

Let us assume that the source terms can be approximated by the parallel momentum sources, i.e., $S_a \simeq [B\langle BS_a^1\rangle/\langle B^2\rangle](v_\parallel f_{a0}/n_a T_a)$. Then our formulation reduces to the conventional neoclassical transport theory in the presence of the parallel momentum sources.

Finally we note that the ion flow velocity is expressed from the momentum balance equation as

$$\boldsymbol{u}_\alpha = \frac{B\,\langle B u_{\parallel\alpha}\rangle}{\langle B^2\rangle}\,\boldsymbol{b} + \frac{c}{e_\alpha n_\alpha m_\alpha B}\,\boldsymbol{b}\times\boldsymbol{S}^1_{\perp\alpha},$$

where $\boldsymbol{S}^1_{\perp\alpha}$ is the perpendicular momentum source.

References

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