Bifurcation theory for the L-H transition

W. Weymiens, H.J. de Blank and G.M.D. Hogeweij

FOM Institute for Plasma Physics Rijnhuizen, Association EURATOM-FOM, Partner in the Trilateral Euregio Cluster, PO Box 1207, 3430 BE Nieuwegein, The Netherlands

Introduction

The high (H) confinement mode in tokamaks arises when, for sufficient heating power, P_{LH}, a transport barrier emerges near the last closed magnetic surface, which has the character of a bifurcation exhibiting hysteresis in the heat transport (the back-transition to the low (L) confinement mode happens at a lower heating power than P_{LH}). The limited range of available quantitative experimental tests cannot discriminate between the many proposed models. Instead we use bifurcation theory to examine the transition characteristics of the L-H transition, such as hysteresis and dithering.

L-H transition characteristics of a simplified transport model

The degree of confinement is determined by the 1-dimensional radial transport along the minor radius of the tokamak (assuming a slab geometry), in which the particle flux is governed by some effective particle diffusion due to the anomalous transport driven by turbulence. The heat transport has two contributions, one is the effective heat conduction and one is the advection due to the particle flux:

$$\frac{\partial n}{\partial t} = -\frac{\partial \Gamma}{\partial r} \qquad \text{with} \quad \Gamma = -D\frac{\partial n}{\partial r} \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) = -\frac{\partial q}{\partial r} \qquad \text{with} \quad q = -\chi n\frac{\partial T}{\partial r} + \frac{3}{2}T\Gamma \tag{2}$$

A change from low confinement to high confinement can therefore be described by a decrease in the anomalous transport coefficients; particle diffusivity, D, and heat conductivity χ.

The anomalous turbulent transport can be significantly reduced due to flows in the plasma, because these flows tear apart turbulent eddies into smaller ones [3]. There are quite some different physical effects influencing the flow in a plasma, the main effect is probably due to electric fields [4] which, in combination with the always present magnetic field, drives an $(\mathbf{E}\times\mathbf{B})$-drift. So a radial electric field induces a poloidal flow in the plasma, which will reduce the radial extend of the turbulent eddies and therefore also the radial transport even down to neoclassical levels. The dependence of the transport coefficients on the normalized radial electric field, $Z = \frac{v_\parallel}{v_i} = \frac{\rho_p e E_r}{T_i}$, is indicated in Figure 1b. For the evolution of the radial electric field we follow [1,2,4]:

$$\varepsilon\frac{\partial Z}{\partial t} = -G(Z) + c\frac{T}{n}\left(\frac{1}{n}\frac{\partial n}{\partial r} + \frac{\gamma}{T}\frac{\partial T}{\partial r}\right) + \frac{\mu}{\Omega_i}\frac{\partial^2 Z}{\partial r^2} \tag{3}$$

where μ is a viscosity causing the radial transport of poloidal momentum, γ is a factor of order unity indicating the relative influence of the neoclassical contributions of the temperature and

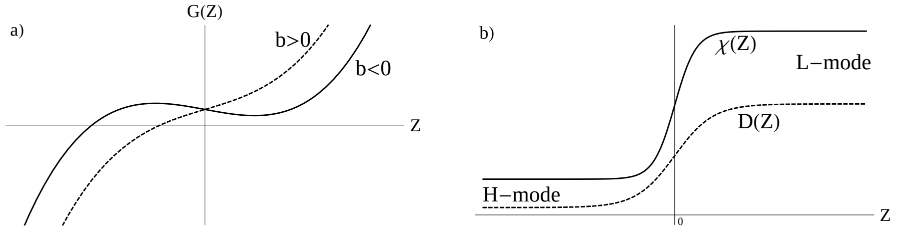

Figure 1: a) Taylor expansion of any nonlinear self-interaction of the radial electric field: $G = a + bZ + Z^3$. b) The dependence of the transport coefficients (particle diffusivity, D, and heat conductivity, χ) on the normalized radial electric field, Z. However, neither the exact shape of the curve is important, nor the relative sizes of the minima and maxima as long as there is some significant difference between the minimum and maximum values and that the transition occurs for both transport coefficients around the same value of Z.

density gradients to the plasma rotation. $G(Z) = a + bZ + Z^3$, but can be any kind of nonlinear function of the radial electric field with the local property as indicated in Figure 1a, with dimensionless constants a and b. The timescale for the electric field evolution is set by the parameter $\varepsilon \sim \mathscr{O}(\Omega_i^{-1})$, with Ω_i the ion cyclotron frequency. Due to the magnitude of the ion cyclotron frequency, both terms of equation (3) proportional to the inverse of it can mostly be neglected except for sudden jumps in time, at which the time derivative of Z becomes large, and sudden jumps in space where the spatial derivatives become important. This last term must be taken into account right at the position of the transition between the normal L-mode transport in the core of the plasma and the reduced transport in the edge transport barrier (i.e. H-mode transport). The spatial region under consideration must be considerably larger than the size of the transport barrier to exclude boundary effects, but small enough that the core boundary stays away from the particle and heat sources in the core of the plasma. The outer edge of the plasma at the scrape-off layer side is fixed at $r = 0$. The inner boundary of the considered spatial region is located at $r = -\infty$, this is allowed because compared to the size of the transport barrier the inner boundary is very far away. The following boundary conditions can therefore be assumed:

$$\left.\frac{1}{T}\frac{\partial T}{\partial r}\right|_{r=0} = \frac{-1}{\lambda_T} \quad , \quad \left.\frac{1}{n}\frac{\partial n}{\partial r}\right|_{r=0} = \frac{-1}{\lambda_n} \qquad \text{and} \qquad \begin{array}{c} \Gamma(r=-\infty) = \Gamma_{core} \\ q(r=-\infty) = q_{core} \end{array} \tag{4}$$

with constant gradient lengths λ_T and λ_n at the edge of the plasma, and constant fluxes, Γ_{core} and q_{core}, arriving from the core of the plasma. In steady state these core fluxes are constant over the entire spatial region, so they must additionally match the boundary conditions at the edge. So the edge steady state of the plasma is restricted to:

$$\frac{D(Z)}{1 + \frac{2}{3}\frac{\chi}{D}\frac{\lambda_n}{\lambda_T}} = -\theta G(Z) \qquad \text{with} \quad \theta = \frac{3}{2}\frac{\Gamma_{core}^2 \lambda_n^2}{c q_{core}} \frac{1}{1 + \gamma \frac{\lambda_n}{\lambda_T}} \tag{5}$$

The control parameter θ dictates the state of the edge plasma, as visualized in Fig. 2. In part a) the hysteresis behaviour is obvious because $\theta_{L-H} \neq \theta_{H-L}$. The size of the hysteresis can be varied, for instance due to the change of the b parameter, until $\theta_{L-H} = \theta_{H-L}$ (see part b)) and only smooth transitions are left.

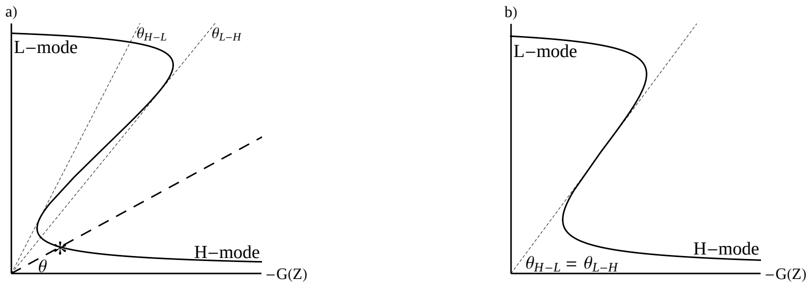

Figure 2: a) Visualization of stationary solution of edge boundary. The solid curve represents the LHS of equation (5), and the dashed line represents the RHS of equation (5). The intersection of the two lines dictates the stationary state (L-mode or H-mode) at the edge of the plasma and depends on the slope, θ, which is therefore one of the control parameters for the L-H transition. b) The two fold bifurcation, causing a jump to and from the H-mode at a slope of θ_{L-H} and θ_{H-L} respectively, merge into a cusp bifurcation.

L-H transitions: a new approach

The L-H transition characteristics in the considered transport model found with the analysis from the previous section, could be found much easier and more complete with the use of bifurcation theory. Bifurcation theory is the analysis of topological changes in the behaviour of dynamical systems. All general L-H transition characteristics are recognized as such topological changes. (I) The sharp transition from L-mode to H-mode when a threshold value of the input power is reached, corresponds to the disappearance of a stationary state of the fusion plasma (L-mode) such that the system must immediately transit to a new stationary state (H-mode), see the red curve in Fig. 3. This disappearance of a stationary state of a dynamical system is called a *fold bifurcation*. (II) The sharp back transition from the H-mode back to the L-mode occurs at a lower threshold value of the input power (orange curve in Fig. 3), this hysteresis behaviour corresponds to two different fold bifurcations in the dynamical system. The point from which these two fold bifurcations start to appear is called a *cusp bifurcation*. (III) Under some plasma conditions oscillatory behaviour is observed before the system settles into the H-mode, these are called 'Dithering' transitions. These oscillations, occuring in the green area of Fig. 3, correspond to limit cycle solutions of the dynamical system occurring due to a *Hopf bifurcation*.

If these three bifurcations are present in a certain dynamical model it automatically implies that all the L-H transition characteristics are present. Fortunately, these three bifurcations are additionally organized by another bifurcation, a so-called *bifurcation of co-dimension 3*. At this point in parameter

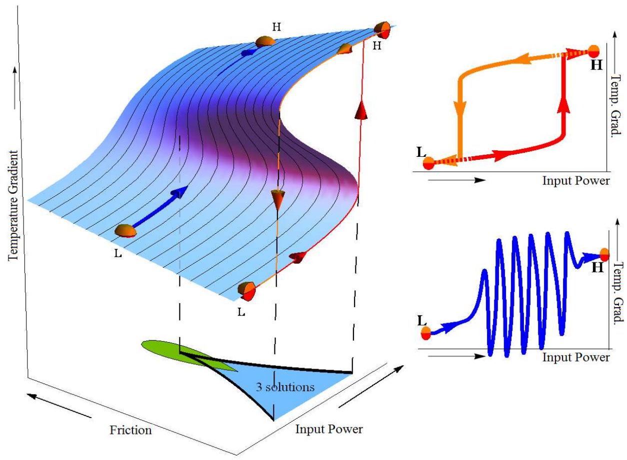

Figure 3: Left: Cusp bifurcation in the edge ∇T as a function of two control parameters (e.g. friction with neutrals and input power). Whether the L-H transition exhibits hysteresis (top right) or oscillations (dithering, bottom right), depends on e.g. the relative effects of heat and particle transport.

space all three bifurcations (i.e. two folds and one Hopf bifurcation) merge together; therefore it is sufficient to determine the presence of this special bifurcation of co-dimension 3 to prove that all the essential transition characteristics are present in the model. The transition criteria described in Fig. 2a indeed correspond to two different fold bifurcations of the entire dynamical system evaluated at the edge of the plasma, see eq. (6). These two fold bifurcations merging together in Fig. 2b correspond to the cusp bifurcation, see eq. (7). The most important benefit of using bifurcation theory is that we found the intersection of this cusp bifurcation with the Hopf bifurcation constructing the organizing center of all the transition characteristics (hysteresis and dithering) of this model, namely the co-dimension 3 bifurcation:

$$\begin{cases} \frac{d}{dZ}\left[\frac{G}{D}\left(1+\frac{2}{3}\frac{\lambda_n}{\lambda_T}\frac{\chi}{D}\right)\right] = 0 & (6) \\ \frac{d^2}{dZ^2}\left[\frac{G}{D}\left(1+\frac{2}{3}\frac{\lambda_n}{\lambda_T}\frac{\chi}{D}\right)\right] = 0 & (7) \\ \frac{\chi_L}{D_L}-\frac{3}{2}\gamma = 0 & (8) \end{cases}$$

Conclusion

By finding the co-dimension 3 bifurcation (eqs. 6–8) in the parameter space of the transport model (eqs. 1–3) it can immediately be concluded that all essential transition characteristics are present in this model. Additionally, the exact form of the L-H control parameter θ indicates that not only increasing the heat flux brings the edge state of the system towards H-mode, but also other plasma parameters can be used to control the L-H transition, e.g. particle flux and the gradient lengths. Furthermore, the co-dimension 3 bifurcation in this transport model dictates that for some plasma conditions oscillatory L-H transitions must occur. As indicated by the green area in Fig. 3 which covers the cusp bifurcation, such that in the dither regime the cusp type behaviour cannot be observed anymore. This is indeed confirmed by the analysis of the limit case where γ becomes very small, in which only oscillatory solutions where found at the transition from L-mode to H-mode [2]. Physically this corresponds to the limit where the temperature gradient does not have much influence on the evolution of the radial electric field and the dynamics is dominated by the density gradient. Vise versa, it can be concluded that the temperature gradient and heat flux influence on the radial electric field are essentially necessary for the sharp forward and backward transitions and the hysteresis behaviour.

References

[1] H. Zohm, *et al.*, Phys. Rev. Lett. **72** (1994) p222

[2] S.I. Itoh *et al.*, Phys. Rev. Lett. **67** (1991) p2485

[3] K. H. Burrell, Phys. Plasmas **4** (1997) p1499

[4] K. Itoh and S-I Itoh, Plasma Phys. Control. Fusion **38** (1996) p1

Acknowledgements This work, supported by the European Communities under the contract of Association between EURATOM/FOM, was carried out within the framework of the European Fusion Programme with financial support from NWO. The views and opinions expressed herein do not necessarily reflect those of the European Commission. This work is supported by NWO-RFBR Centre-of-Excellence on Fusion Physics and Technology (Grant nr. 047.018.002)