An efficient numerical method for gyrokinetic δf Vlasov simulations in helical plasmas

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Three-dimensional configurations of helical plasmas bring numerical difficulties to gyrokinetic simulations of plasma turbulent transport in toroidal fusion devices. Watanabe *et al.* have investigated the ion-temperature-gradient (ITG)-driven turbulence in helical plasmas by means of the gyrokinetic Vlasov simulation code GVK [1]. Because of a large number of grid points along a magnetic field line for resolving particle motions in helical ripples, however, the Courant-Friedrichs-Lewy (CFL) condition on the parallel advection term as well as the perpendicular magnetic drift term severely restricts time steps in the linear limit. To address this problem, we propose an efficient numerical method for linear analysis of instabilities in helical plasmas by means of gyrokinetic δf Vlasov simulations, employing semi-Lagrangian and additive semi-implicit Runge-Kutta schemes. The new scheme is free from the CFL restrictions for the linear terms.

Gyrokinetic equations

Employing the flux tube model, the perturbed ion gyrocenter distribution function $f_{\boldsymbol{k}}(z, v_\parallel, \mu)$ is represented in the perpendicular wave number space $\boldsymbol{k} = (k_x, k_y)$. The linearized gyrokinetic Vlasov equation for $f_{\boldsymbol{k}}$ in the electrostatic limit is given by

$$\left[\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + i\omega_\mathrm{d} - \frac{\mu \nabla_\parallel B}{m_\mathrm{i}} \frac{\partial}{\partial v_\parallel}\right] f_{\boldsymbol{k}} = -\frac{eF_\mathrm{M}}{T_\mathrm{i}} \left[v_\parallel \nabla_\parallel + i\omega_\mathrm{d} - i\omega_*\right] \Phi_{\boldsymbol{k}} + \mathscr{C}(f_{\boldsymbol{k}}), \qquad (1)$$

where B, m_i, e, F_M, T_i, ω_d, and ω_* are the magnetic field strength, ion mass, elementary charge, Maxwellian distribution function, ion temperature, magnetic and diamagnetic drift frequencies, respectively. The parallel velocity $v_\parallel$ and the magnetic moment μ are employed as the velocity space coordinates. The parallel gradient is written as $\nabla_\parallel = (1/q_0 R_0)\partial_z$, where q_0, R_0, and z are the safety factor, major radius, and field-aligned coordinate, respectively. The gyrocenter electrostatic potential $\Phi_{\boldsymbol{k}}$ is related to the electrostatic potential $\phi_{\boldsymbol{k}}$ by $\Phi_{\boldsymbol{k}} = J_0(k_\perp \rho_\mathrm{i})\phi_{\boldsymbol{k}}$, where J_0 and ρ_i are the zeroth-order Bessel function and ion Larmor radius, respectively. The electrostatic potential is determined by the quasi-neutrality condition with an adiabatic electron response. We employ the gyrophase-averaged Lenard-Bernstein operator as the collision operator $\mathscr{C}$.

The magnetic field strength in a large-aspect-ratio helical plasma is approximated as

$$B = B_0 \left\{ 1 - \varepsilon_{00} - \varepsilon_{\rm t} \cos z - \sum_{l=L-1}^{l=L+1} \varepsilon_l \cos[(l - Mq_0)z - M\alpha] \right\}, \tag{2}$$

where L and M are the poloidal and toroidal period numbers of the main component of the helical field. Here, we choose the field line label of $\alpha = 0$. The parameters ε_{00}, $\varepsilon_{\rm t}$, ε_L and $\varepsilon_{L\pm1}$ correspond to the averaged magnetic hill/well, toroidicity, main helical field and two sideband components, respectively [1].

Hybrid method of semi-Lagrangian and additive semi-implicit Runge-Kutta schemes

By means of the operator splitting method [2], Eq. (1) can be split into the parallel motions and the others:

$$\frac{\partial f_{\boldsymbol{k}}}{\partial t} = \{H, f_{\boldsymbol{k}}\}_\parallel , \tag{3}$$

$$\frac{\partial f_{\boldsymbol{k}}}{\partial t} = -i\omega_{\rm d} f_{\boldsymbol{k}} - \frac{eF_{\rm M}}{T_{\rm i}} \left[v_\parallel \nabla_\parallel + i\omega_{\rm d} - i\omega_* \right] \Phi_{\boldsymbol{k}} + \mathscr{C}(f_{\boldsymbol{k}}), \tag{4}$$

where $H = v_\parallel^2/2 + \mu B/m_{\rm i}$ is the particle kinetic energy per unit mass and $\{f,g\}_\parallel = \nabla_\parallel f \partial_{v_\parallel} g - \partial_{v_\parallel} f \nabla_\parallel g$ denotes the parallel Poisson brackets.

Equation (3) represents the field-aligned advection of $f_{\boldsymbol{k}}$ characterized by the parallel particle motions of $ds/dt = \{s, H\}_\parallel$ with $s = (qR_0 z, v_\parallel)$. We further split Eq. (3) into two linear advection equations in z and $v_\parallel$, *i.e.*, $\partial f_{\boldsymbol{k}}/\partial t = -v_\parallel \nabla_\parallel f_{\boldsymbol{k}}$ and $\partial f_{\boldsymbol{k}}/\partial t = (\mu \nabla_\parallel B/m_{\rm i}) \partial_{v_\parallel} f_{\boldsymbol{k}}$. Their analytical solutions are $f_{\boldsymbol{k}}(s, t+\Delta t) = f_{\boldsymbol{k}}(s - u\Delta t, t)$, where $s = qR_0 z$ or $v_\parallel$ and $u = v_\parallel$ or $-\mu \nabla_\parallel B/m_{\rm i}$, and computed by a semi-Lagrangian scheme [3]. Practically, one has to evaluate the value of $f_{\boldsymbol{k}}(s - u\Delta t, t)$ by using one-dimensional interpolations. For more details, see Ref. [4].

On the other hand, Eq. (4) is regarded as an additive operator of the perpendicular drift, source and collision terms. In order to solve Eq. (4), we employ additive semi-implicit Runge-Kutta schemes (ASIRK) [5] and treat the magnetic drift term implicitly. Since the coefficient matrix of the magnetic drift term is diagonal, one can easily compute its semi-implicit time integration without using matrix solvers.

Parallel dynamics in helical plasmas

The parallel dynamics, Eq. (3), is computed by a semi-Lagrangian scheme with a second-order operator splitting method. We employ 192×256 grid points and a time step size $\Delta t/t_{\rm tr} = 0.1$, where $t_{\rm tr} = L_n/v_{\rm ti}$ with the density scale length L_n and the ion thermal velocity $v_{\rm ti}$.The initial profile is given by $f(z, v_\parallel, t=0) = F_{\rm M}(H)(1+\cos z)$, and the periodic boundary condition is employed in z. Characteristic curves of the dynamics are given as contour lines of the particle kinetic energy H, as shown in Fig. 1 (a). There are trajectories of helical-ripple-trapped particles

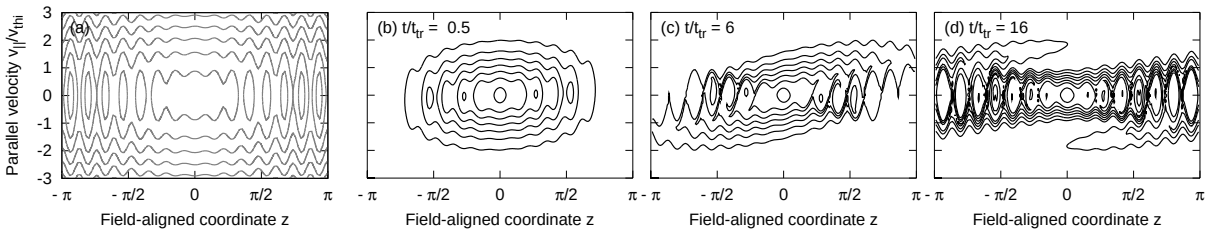

Figure 1: (a) Contour lines of the particle kinetic energy H with the helical field of Eq. (2), and (b)-(d) snapshots of equi-contours of the distribution function $f(z, v_{\|})$ in parallel phase space (where $L = 2, M = 10, q_0 = 1.7, \varepsilon_{00} = 0, \varepsilon_{\rm t} = 0.087, \varepsilon_{L-1} = -0.74, \varepsilon_L = 1.2, \varepsilon_{L+1} = -0.24$ and $\mu B_0/T_{\rm i} = 4.0$). Horizontal and vertical axes are defined by z and $v_{\|}$, respectively.

as well as those of passing particles. Snapshots of the contour lines of the distribution functions are shown in Fig. 1 (b)-(d). The distribution function is advected along the contour lines of the particle kinetic energy. While passing particles elongate the profile, trapped particles stay in the trapped regions. Thus, fine-scale structures appear at the trapped-passing boundary.

To estimate the temporal accuracy, we calculate the norm of the errors $L_{\rm E}$ defined by

$$L_{\rm E} = \sqrt{\sum_{i=1}^{N_z}\sum_{j=1}^{N_v}\left|f_{i,j} - f_{i,j}^{\rm ref}\right|^2 \Big/ \sum_{i=1}^{N_z}\sum_{j=1}^{N_v}\left|f_{i,j}^{\rm ref}\right|^2}, \quad (5)$$

where N_z and N_v are the number of grid points in the z and $v_{\|}$ coordinates, respectively. The reference solution $f_{i,j}^{\rm ref}$ is computed with higher resolutions ($N_z = 768, N_v = 256, \Delta t/t_{\rm tr} = 0.001$). Figure 2 plots $L_{\rm E}$ as a function of the time step size. The Δt dependence of the error has a plateau for $\Delta t/t_{\rm tr} < 0.5$, while $L_{\rm E}$ monotonically increases for $\Delta t/t_{\rm tr} > 0.5$. According to the figure, the presented scheme is accurate to the second order in time, as expected. It is found that the temporal errors dominate when the time step size is comparable to the transit time of passing particles through one helical ripple, *i.e.*, $\Delta t \sim L_{\rm rip}/v_{\|\max} \sim 0.4 t_{\rm tr}$ for the parameters employed here.

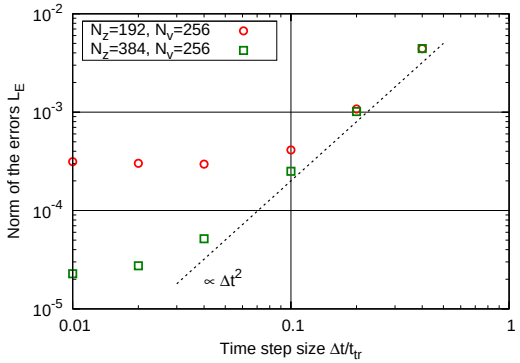

Figure 2: Norms of errors $L_{\rm E}$ at $t/t_{\rm tr} = 2.0$ as a function of the time step size Δt for $N_z = 192, 384$ and $N_v = 256$.

Linear ion-temperature-gradient instabilities in helical plasmas

Employing the hybrid method of semi-Lagrangian and additive semi-implicit Runge-Kutta schemes (SLASIRK), we have carried out linear ITG simulations of a helical plasma. Physical and numerical settings are the same as those for the inward-shifted LHD case shown in Ref. [1].

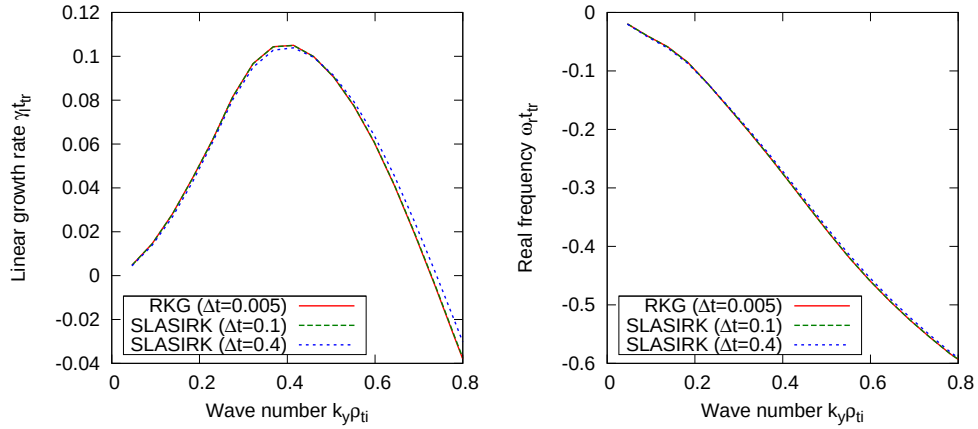

Figure 3: (a) Linear growth rates γ and (b) real frequencies $\omega_{\rm r}$ as a function of the poloidal wave number $k_y\rho_{\rm ti}$. The solid, dashed and dotted lines represent the results obtained by RKG with $\Delta t/t_{\rm tr} = 0.005$ and SLASIRK with $\Delta t/t_{\rm tr} = 0.1, 0.4$, respectively.

The linear growth rates and the real frequencies are plotted as a function of the poloidal wave number k_y in Fig. 3. The results agree well with the results obtained by using the fourth-order Runge-Kutta-Gill scheme (RKG), while taking the time step size larger than that of RKG. The presented numerical method gives sufficiently accurate results for $\Delta t/t_{\rm tr} < 0.4$, which is comparable to the transit time of passing particles through one helical ripple.

Summary

A new method for solving gyrokinetic equation in a flux tube geometry is developed. First, we have applied the semi-Lagrangian scheme to the parallel dynamics in helical plasmas. It is demonstrated that the time step size is free from the CFL conditions and is restricted by a physical time, *i.e.*, the transit time through the magnetic ripples. Second, a hybrid method of semi-Lagrangian and additive semi-implicit Runge-Kutta schemes is applied to linear computations of the ITG instability in helical plasmas, and confirm that a stable and efficient computation is possible with time steps much longer than the CFL conditions for all linear terms.

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