

## Helically symmetric magnetic islands: equilibrium and stability

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The MHD\_NX code computes the ideal MHD stability of plasma equilibrium configurations with arbitrary topology of magnetic surfaces, including doublets, reversed current tokamak configurations with magnetic islands (current holes and AC operation) and multi-connected plasmas like two separated plasma columns (droplet) surrounded by a vacuum region and a conducting wall [1–5]. The code uses unstructured triangular grids with a possibility of adaptation to the magnetic surfaces and sharp solution features.

The formulation of equilibrium and ideal MHD stability problems with helical symmetry allows one to model configurations with single helicity magnetic islands. The results of the modeling related to tokamak stability in the presence of islands, the ways to improve the numerical approximation of the ideal MHD condition  $E \cdot B = 0$  and adaptive gridding in ideal MHD stability problems are discussed.

### 1 Helically symmetric equilibria

A simple model for single helicity islands in tokamak is a helically symmetric equilibrium in which boundary perturbations give rise to magnetic islands. The helical flux function  $\psi_h$  can be found as a solution of the generalized Grad-Shafranov equation [6]:

$$-\nabla \cdot \left( \frac{\nabla \psi_h}{g_{33}} \right) = p' + \frac{f f'}{g_{33}} - f \nabla \cdot \left( \frac{e_3 \times e^3}{g_{33}} \right). \quad (1)$$

The equilibrium magnetic field  $\vec{B} = (\nabla \psi_h \times e_3 + f e_3)/g_{33}$  is represented using the curvilinear coordinates  $(x^1, x^2, x^3) = (u, v, z)$  with the corresponding covariant vectors  $e_k = \partial \vec{r} / \partial x^k$  and contravariant  $e^k = \nabla x^k$  vectors. The  $(u, v) = (r \cos \theta_h, r \sin \theta_h)$  plane is rotating about the origin according to the polar coordinate transformation  $(r, \theta)$  into  $(r, \theta_h = \theta - \kappa z)$ , where the helix pitch is  $2\pi/\kappa$ . The metric tensor of the chosen coordinate system  $(u, v, z)$  is as follows:

$$g_{ik} = \begin{vmatrix} 1 & 0 & -\kappa v \\ 0 & 1 & \kappa u \\ -\kappa v & \kappa u & 1 + \kappa^2(u^2 + v^2) \end{vmatrix} = \begin{vmatrix} 1 + \kappa^2 v^2 & -\kappa^2 uv & \kappa v \\ -\kappa^2 uv & 1 + \kappa^2 u^2 & -\kappa u \\ \kappa v & -\kappa u & 1 \end{vmatrix}^{-1}. \quad (2)$$

The Jacobian of the coordinate system is  $\det g_{ik} = 1$ .

## 2 Ideal MHD stability: the problem formulation for helical symmetry

For the stability analysis the potential and kinetic energy functionals can be expressed in terms of the electric field perturbation  $\vec{E} = i\omega\vec{e}$ ,  $\vec{e} = -\vec{\xi} \times \vec{B}$  (time dependence  $e^{i\omega t}$  is assumed for the eigenvalue problem):

$$W_p = \frac{1}{2} \int \left\{ |\nabla \times \vec{e}|^2 - \frac{\vec{j} \cdot \vec{B}}{B^2} \vec{e} \cdot \nabla \times \vec{e} + \frac{\vec{j} \cdot \vec{e}}{B^2} [2\vec{B} \cdot \nabla \times \vec{e} - \vec{t} \cdot \vec{e}] \right\} d^3r, \quad (3)$$

$$K_p = \frac{1}{2} \int \rho |\vec{e}|^2 / B^2 d^3r, \quad \vec{t} = \vec{j} + B^2 \nabla \left( \frac{1}{B^2} \right) \times \vec{B}, \quad (4)$$

combined with the requirement  $(\vec{e} \cdot \vec{B}) = 0$ . The surrounding vacuum region (free boundary) can also be taken into account (see [1]).

For the force-free case the last term in the functional (3) vanishes because  $\vec{j} = \vec{j}_{||} + \vec{B} \times \nabla p / B^2$ ,  $\vec{j}_{||} = (\vec{j} \cdot \vec{B}) / B^2 \vec{B}$ , so that  $\vec{j} \cdot \vec{e} = 0$  for  $\nabla p = 0$ . In this paper we consider only force-free helical equilibria.

The approach to approximate and solve the stability problem on triangular grids is the same as for the axisymmetric plasmas [3] but with the basis functions defined on an unstructured grid in the  $(u, v)$  plane and with the unknown vector represented as  $\vec{e} = e_z \nabla z + \vec{e}_{pol}$ .

For the harmonics  $e^{imhz} \vec{e}_n$  the  $\nabla \times$  operator writes:

$$\nabla \times \vec{e} = e^{imhz} (\nabla e_{n,z} \times \nabla z + in_h \nabla z \times \vec{e}_{n,pol} + \nabla \times \vec{e}_{n,pol}).$$

Let us note that the coordinate system  $(u, v, z)$  is not orthogonal, so some additional terms need to be taken into account compared to the case of toroidal symmetry.

For  $n \neq 0$  the harmonic amplitude  $\vec{e}_n$  becomes complex and a complex matrix solver is needed. In the MHD\_NX code the direct solver from the PETSc package is used, which is available both in real and complex versions.

## 3 Numerical results

A standard simple model for tokamak in the limit of large aspect ratio is one dimensional circular cylinder equilibrium with the safety factor  $q = rB_z / RB_\theta$ , where  $a$  is the minor radius of plasma,  $0 < z < 2\pi R$ ,  $R$  is the major radius of an equivalent torus with aspect ratio  $R/a \gg 1$ .

Exact stability criterion is known for the circular cylinder with piece-wise constant current density [7]:

$$j_z(r) = J_1 + J_2, \quad 0 \leq r \leq a_0, \quad j_z(r) = J_2, \quad a_0 \leq r \leq a. \quad (5)$$

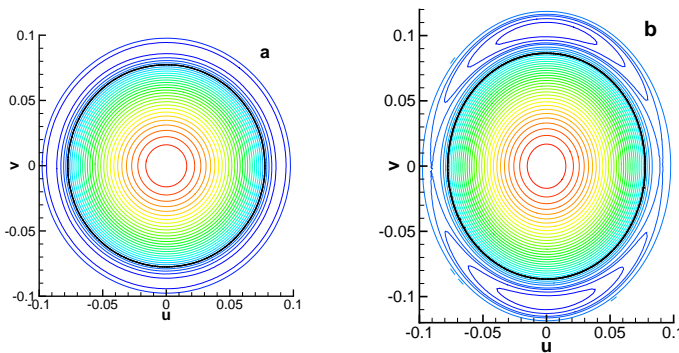
It gives the profile of the rotational transform  $\iota = 1/q$  with the values  $\iota_0 = (J_1 + J_2)/2$  and  $\iota_a = (J_1(a_0/a)^2 + J_2)/2$  at the magnetic axis and the plasma boundary, respectively.

Assuming a conducting wall at infinity, for the poloidal and toroidal wave numbers  $m$  and  $n$  the criterion of the external kink mode instability with  $m\iota_a > n$  is given by the inequality:

$$(m\iota_a - n - J_2/2)(m\iota_0 - n - J_1/2) - J_1 J_2 (a_0/a)^{2m} / 4 < 0. \quad (6)$$

For modeling the reference configuration is chosen with  $(a_0/a)^2 = 0.6$ ,  $q_0 = 1.2$ ,  $q_a = 1.8$  that corresponds to the following values of the constants  $J_1 = 1.39$ ,  $J_2 = 0.2J_1 = 0.278$ . It is strongly unstable against the mode  $m/n = 2/1$ . The minor radius value was set to  $a = 0.1$  with the major radius  $R = 1$  and magnetic field  $B_z = 1$  to satisfy the limit of large aspect ratio.

The same cylindrically symmetric equilibrium can be described by the equilibrium equation (1) with arbitrary choice of  $\kappa$ . This can be taken e.g. as  $\kappa = 0$  (cylinder case) or with some  $R\kappa = n_z/n_\theta$  corresponding to a resonant value  $q = n_\theta/n_z$  of the safety factor inside the plasma. In the large aspect ratio limit the helical flux  $\psi_h = \psi_{cyl} - B_z\kappa(a^2 - r^2)/2$  gives  $\nabla\psi_h = 0$  at the resonant magnetic surface  $q = 1/(R\kappa)$ . For the  $R\kappa = 2/3$  the rational surface  $q = 1.5$  becomes topologically unstable under perturbations of the plasma boundary given in the  $(u, v)$  plane (level lines of the solution (1) are shown in Fig.1a). Elongation of the plasma cross-section gives rise to the  $m = 2$  magnetic islands (Fig.1b).



*Fig.1. Helical flux level lines for 1D circular cylinder (a) and 2D case with plasma cross section elongation  $E = 1.2$  (b). Black lines shows the delimiting line passing through the point  $(a_0/a)^2 = 0.6$ .*

Let us note that choosing different values of  $R$  does not change the solution for the cylindrical flux  $\psi_{cyl}$  under fixed  $j_z$  profile but do modify the helical flux under chosen constant  $B_z$  and  $R\kappa$ . It corresponds to the change of flux through the helical surface in the equivalent tori with different major radii. On the other hand, it provides a series of equivalent equilibria to test both equilibrium and stability codes in one-dimensional cylindrical case.

The solution of the helical equilibrium equation is obtained using the Matlab pde tool toolbox. In case of the two step current density profile (5) under approximation that in the last term of (1)  $f = const$  the procedure is as follows: the boundary of the plasma in the  $(u, v)$  plane is analytically prescribed as an ellipse with chosen elongation, a triangular unstructured grid is generated by the prescribed grid edge length inside the boundary, pdenonlin with user defined right hand side is employed to solve the nonlinear equation. The main nonlinearity is in the mapping of the "delimiting" line  $\psi_h = \psi_{h0}$  with the value of  $\psi_{h0}$  determined in the middle of the mesh cell closest to the prescribed limiter point at  $(u, v) = (a_0, 0)$  (Fig.1b). Then the value of the two-step current density  $j_3 = ff'$  at the grid is determined by the local value of  $\psi_h$ . The computed equilibria and the position of the delimiting line are shown in Fig.1.

The stability computations were performed with the MHD\_NX code modified for the force-free helical symmetry geometry. Fig.2 shows the comparison of the results of the test computations of the external  $m = 2$  mode growth rates for different values of  $nq_a$  in the straight cylinder,

where  $n$  is "toroidal" wave number for the harmonic  $e^{inz/R}$ . Let us note that in the helical case several cylindrical modes can become unstable for a single chosen wave number  $n_h$  because of coupling of the resonant modes with different resonant toroidal wave numbers  $n$  through the helicity of the coordinate system. Moreover, the correspondence between the helical and cylindrical wave numbers needs to be taken into account:  $n_h = n/R - m\kappa$  to trace the chosen mode in the helical spectrum. The growth rates of the external kink modes in the equilibria with  $m = 2$  islands with  $R\kappa = 2/3$  helicity are just slightly modified. However the  $n_h = 0$  modes become unstable once the islands open (Fig.3).

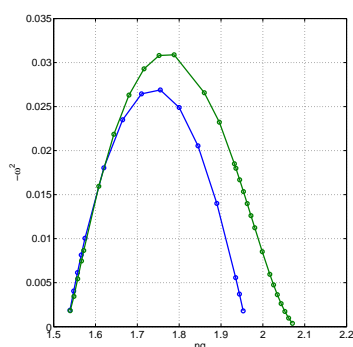


Fig.2. Straight cylinder test: the external  $m = 2$  mode growth rates numerically reproduced for conventional (blue) and "helical" (green) configurations with conducting wall at  $r_w = 2a$ . Total number of grid nodes is  $N = 4600$ .

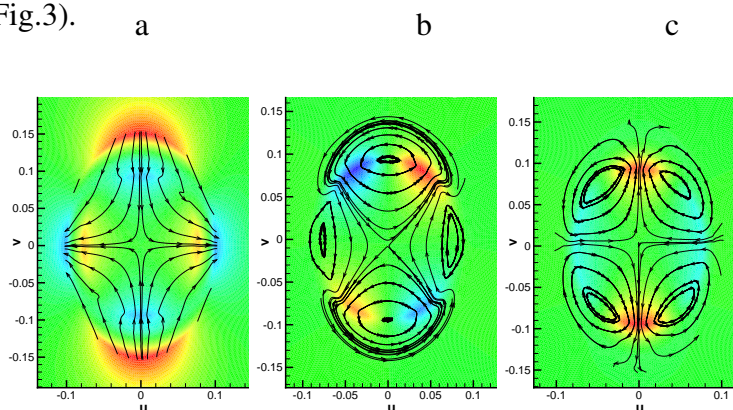


Fig.3. External  $n = 1$  kink mode structure for the helical equilibrium with the elongation  $E = 1.5$ :  $\omega^2 = -0.0183$  (a). Unstable modes with  $n_h = 0$ :  $\omega^2 = -0.0299$  (multiplicity 2) (b),  $\omega^2 = -0.0122$  (c). Total number of nodes  $N = 28471$ . The contour plots of  $e_z = \vec{\xi} \cdot \nabla \psi_h$  and the streamlines for  $\vec{e}_{pol}$  are shown.

#### 4 Discussion

The formulations of equilibrium and ideal MHD stability problems on unstructured grids for plasmas with helical symmetry were developed. Helically symmetric equilibria equivalent to cylindrically symmetric configurations and with magnetic islands induced by helical plasma boundary shaping were computed. The helically symmetric plasma option was implemented into the MHD\_NX ideal stability code. Test calculations of cylindrically symmetric plasma stability in different coordinate systems were performed for the constant and two-step current density profile configurations in the cylinder. The ideal stability calculations with helical magnetic islands displayed a weak influence of the islands on the external kink mode stability, but revealed a family of the two-dimensional instabilities winding with the equilibrium islands (helical wave number  $n_h = 0$ ).

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