Ideal MHD stability of helically symmetric magnetic islands

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A new version of the MHD_NX code, that computes the ideal MHD stability of helically symmetric equilibria with arbitrary topology of magnetic surfaces [1], was applied to the investigation of equilibrium magnetic islands in tokamak-like conditions.

Any helical deformation of the plasma boundary shape from a circular cylinder results in the breaking of topology of the helical flux level lines and appearance of magnetic islands at the place of the magnetic surface $q = m/n$ in the original large aspect ratio tokamak equilibrium, provided that the helical pitch is the same for the equilibrium and magnetic lines at the resonant surface. A solution family of the generalized Grad-Shafranov equation with a linear dependence of the source current density on the helical flux was employed to compute equilibria with various chains of islands.

Internal ideal MHD modes resonant to the corresponding island chain (longitudinal wave number $n_h = 0$ in the helical coordinates) are found to be robustly unstable for $m = 1$ and $m = 2$ boundary deformations, while stable for higher plasma shape poloidal harmonics $m \geq 3$.

1 Helically symmetric equilibria with islands

The helical flux function ψ_h can be found as a solution of the generalized Grad-Shafranov equilibrium equation [2]:

$$-\nabla\cdot\left(\frac{\nabla\psi_h}{g_{33}}\right) = p' + \frac{ff'}{g_{33}} - f\nabla\cdot\left(\frac{e_3\times e^3}{g_{33}}\right). \qquad (1)$$

The equilibrium magnetic field $\vec{B} = (\nabla\psi_h \times e_3 + fe_3)/g_{33}$ is represented using the curvilinear coordinates $(x^1, x^2, x^3) = (u, v, z)$ with the corresponding covariant $e_k = \partial\vec{r}/\partial x^k$ and contravariant $e^k = \nabla x^k$ vectors. The $(u, v) = (r\cos\theta_h, r\sin\theta_h)$ plane is rotating about the origin according to the polar coordinate transformation (r, θ) into $(r, \theta_h = \theta - \kappa z)$, where the helix pitch is $2\pi/\kappa$. Then equation (1) takes the form:

$$-\sum_{i,k=1}^{2}\frac{\partial}{\partial x^i}\left(\frac{G_{ik}}{g_{33}}\frac{\partial\psi_h}{\partial x^k}\right) = p' + \frac{ff'}{g_{33}} - \frac{2\kappa f}{g_{33}^2}, \qquad (2)$$

$$G_{11} = 1+\kappa^2v^2,\ G_{12} = -\kappa^2uv,\ G_{22} = 1+\kappa^2u^2,\ g_{33} = 1+\kappa^2(u^2+v^2).$$

We assume a linear dependence in ψ_h for the current density $j_3 = ff' + g_{33}p'$, similarly to the axisymmetric equilibria with reversed current density considered in [3]. For force free configurations ($p' = 0$), we have

$$j_3 = \alpha\psi_h/a^2 + A, \qquad (3)$$

where a is the plasma minor radius and the coefficients α and A are varied to obtain a family of equilibria with islands. A standard simple model for tokamak in the limit of large aspect ratio is 1D circular cylinder equilibrium with the safety factor $q = rB_z/RB_\theta$, where $0 < z < 2\pi R$, R is the major radius of the equivalent torus with aspect ratio $R/a \gg 1$. The values $R = 1$ and

$a = 0.1$ were used with $f = \text{const}$. The helical flux and the poloidal flux in the cylinder are related by $\psi_h = \psi_{cyl} - B_z\kappa(a^2 - r^2)/2$. The values of $\alpha = 21$ and $f = 1$ provide solutions of the equilibrium equation (2), (3) with a local maximum at the main magnetic axis and several local minima of the ψ_h inside the plasma. The local minima correspond to the presence of the magnetic surface $q = 1/(R\kappa)$ inside the 1D cylindrical equilibrium. The value of A controls the boundary current density and the global shear. The equilibrium equation is linear in ψ_h and can be readily solved numerically.

2 Ideal MHD stability with helical islands

The stability computations were performed with the MHD_NX code [1] modified for an arbitrary 2D equilibrium configuration (cylindrical, toroidal axisymmetric and helical symmetry).

The large aspect ratio approximation assumes equilibrium configuration with the period $2\pi R$ in z, R being a major radius of the tokamak. The pitch of the magnetic line at the surface with the safety factor q is $2\pi Rq$. For rational magnetic surfaces $q = m/n$ the plasma boundary deformation with the same helicity opens magnetic islands. If the boundary deformation in helical coordinates possesses the $2\pi/m$ rotational symmetry, the configuration period is $2\pi R/n$, hence the initial period is preserved. Let us note that any shaping of the plasma column including spatial helical axis is admissible to produce single helicity magnetic islands for $q = 1$ and $q = 1/n$ in general. The rotational symmetry $2\pi/m$ is provided by the following plasma boundary representation:

$$u = u_0 + a\rho\cos\tau, \quad v = a\rho\sin\tau, \quad \rho = (1 + h_m\cos m\tau)^{1/2}, \quad 0 < \tau < 2\pi. \tag{4}$$

For $m > 1$ rotational symmetry the shift of the plasma boundary from the origin in the (u, v) plane should be zero, $u_0 = 0$.

To model an external mode with a different helicity from the equilibrium helicity, the following relation between the longitudinal wave number n_h in helical coordinates for the harmonic e^{in_hz} and the corresponding cylindrical wave numbers m, n for $e^{i(nz/R - m\theta)}$ needs to be taken into account:

$$Rn_h = n - mR\kappa. \tag{5}$$

In particular, the relation (5) can be used to check the resonant condition for the m/n external kink mode instability $nq_1 < m$, where q_1 is the safety factor at the boundary in the periodic cylinder with the length $2\pi R$. For the resonant surface $q = 1/(R\kappa) = 3/2$ in the plasma and the external $m/n = 2/1$ mode it results in $Rn_h = -1/3$, meaning that the mode is not periodic for $R = 1$. The integer value of $n_h = -1$ corresponds to $R_s = 1/3$. In large aspect ratio approximation the helical flux ψ_h and the stability properties will be the same for the helical equilibrium scaling: $f_s/R_s = f/R$, $R_s\kappa_s = R\kappa$, $\kappa_s f_s = \kappa f$, keeping the safety factor q unchanged in the equilibrium with different period. The mode structure of the external $n_h = -1$ mode in the helical equilibrium with triangularly deformed cross section and $m = 3$ islands is shown in Fig.1a. The external kink mode $m = 2$ is not significantly modified by the presence of the islands. The difference in the growth rate between the helical and cylindrical cases is about

20%: $\omega^2/\omega_{Ap}^2 = -0.1276$ and $\omega^2/\omega_{Ap}^2 = -0.1565$ respectively. Let us note that the growth rate dependence on the value of κ under fixed ψ_h, ff' and q is quite weak in general due to large aspect ratio.

The spatial helical axis is produced by the shift of the plasma cross section $u_0 > 0$ in (4). It results in the $m = 1$ deformation of the boundary and gives rise to islands at $q = 1/(R\kappa) = 1$. In Fig.1b the level lines of ψ_h in several cross sections of such an equilibrium are shown for the value of $A = 1.75$. Let us note that more peaked current density profile for lower value of $A = 0.75$ in (3) results in the $m = 2$ island chain generation (Fig.1c). Such equilibrium configurations are robustly unstable against the internal kink modes with the same helicity as the equilibrium ($n_h = 0$). The streamlines of the plasma displacement demonstrate the tilt of the two islands with opposite directions of poloidal and longitudinal motions.

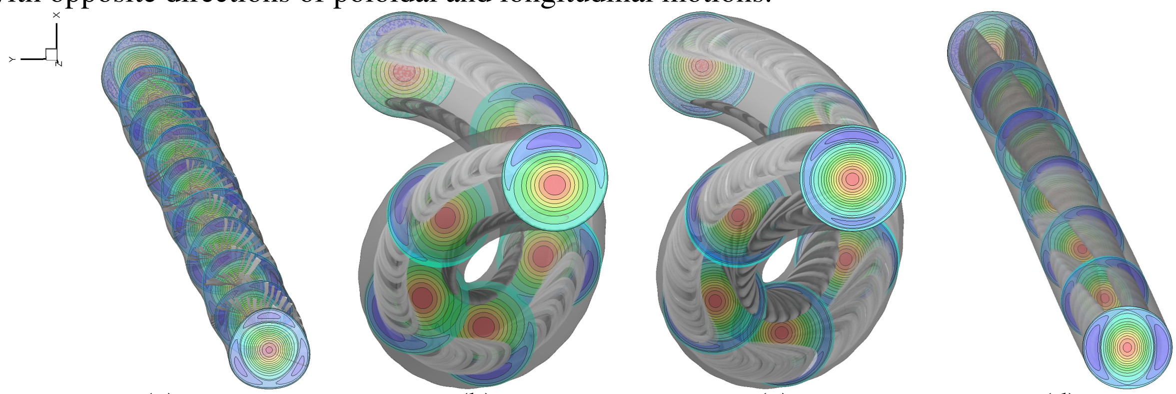
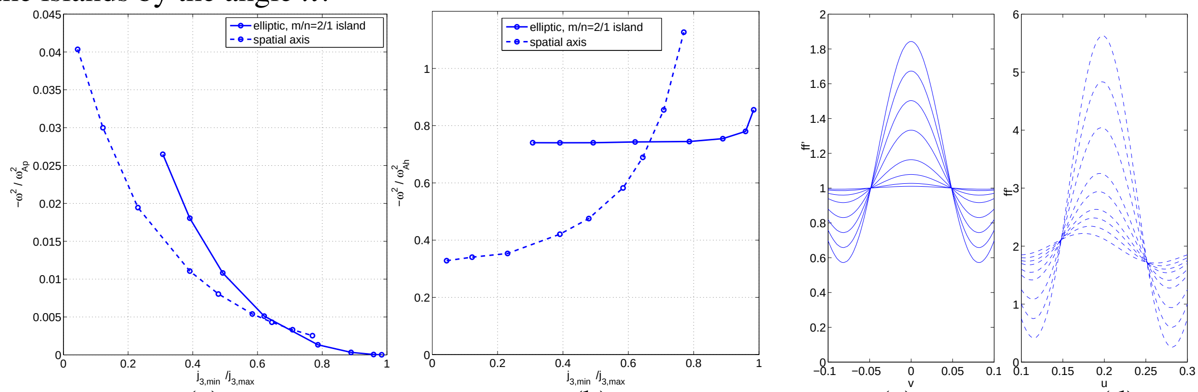


Fig.1. Helical flux function ψ_h contour plots in several cross-sections along z and the streamlines (grey ribbons) of the unstable mode. (a) The island at the $q = 3/2$ surface and the external $m = 2$ mode ($R\kappa = 2/3$, $n_h = -1$, $A = 0.75$, $h_3 = 0.08$, $0.6 < q < 1.8$). $R = 1/3$, three periods are shown. (b) Shifted circular cross section and islands at the $q = 1$ resonant surface, internal mode ($R\kappa = 1$, $n_h = 0$, $A = 1.75$, $u_0 = 0.2$, $0.7 < q < 1.1$). (c) Islands at the $q = 1$ resonant surface with more peaked current density, internal mode ($R\kappa = 1$, $A = 0.75$, $0.3 < q < 1.3$). (d) Islands at the $q = 2$ surface, internal mode ($R\kappa = 1/2$, $A = 0.75$, $h_2 = -0.08$, $1.1 < q < 2.2$).

The findings about the stability with the longitudinal wave number $n_h = 0$ in the helical coordinates can be summarized as follows: the internal modes with the same helicity as the island chain can be unstable due to the presence of $m = 1$ and $m = 2$ islands, while stable for higher order island chains $m \geq 3$. The equilibrium and streamlines of the instability for the equilibrium with elliptical cross section and the $m = 2$ islands at the $q = 2$ magnetic surface are shown in Fig.1d. The displacement structure is very similar to the cases with spatial magnetic axis.

The Fig.2 demonstrates the effect of the current density profile on the stability of the $n_h = 0$ modes. The squared growth rates normalized by the poloidal Alfvén frequency $\omega_{Ap}^2 = (\psi_{cyl,max} - \psi_{cyl,min})^2/(a^4\rho) \approx B_p^2/(\mu_0 a^2 \rho)$ are plotted versus the $j_{3,min}/j_{3,max}$ ratio of the longitudinal current density varied with the parameter A in (3) for the elliptic cross section ($R\kappa = 1/2$) and spatial helical axis ($R\kappa = 1$) equilibria. The growth rate of the instability normalized by ω_{Ap}^2 decreases for flatter profiles in both cases (Fig.2a). However, normalization by the helical Alfvén frequency $\omega_{Ah}^2 = (\psi_{h,max} - \psi_{h,min})^2/(a^4\rho)$, defined through the helical flux ψ_h, reveals that the growth rate is almost unchanged for the case of elliptic cross section, but increases for flatter current density profiles in the case with spatial helical axis. The weak dependence of the instability growth rate on the current density profile corresponds to weak variations of the topology and size of the $m/n = 2/1$ islands in the equilibria with elliptic cross section in contrast to the spatial axis case for which the topology of islands change. The profiles of ff' are plotted in Fig.2c and Fig.2d for both cases. Further increase of the parameter A ($A > 1$ for $R\kappa = 1/2$ and $A > 1.9$ for $R\kappa = 1$) leads to hollow current density profiles and to the change of the phase of the islands by the angle π.



(a) (b) (c) (d)

Fig.2. (a) The squared growth rates of the $n_h = 0$ modes normalized by the ω_{Ap}^2 versus the min/max ratio for j_3: $R\kappa = 1/2$, $h_2 = -0.08$ (solid lines), $R\kappa = 1$, $u_0 = 0.2$ (dashed lines). (b) The same growth rates normalized by ω_{Ah}^2. (c) The $j_3 = ff'$ profiles versus v at $u = 0$ (across the islands) for the $R\kappa = 1/2$ equilibria corresponding to (a) and (b). (d) The $j_3 = ff'$ profiles versus u at $v = 0$ (across the islands) for the $R\kappa = 1$ equilibria.

3 Discussion

Free boundary ideal MHD stability computations of 2D equilibria with helical magnetic islands confirmed the weak influence of non-resonant magnetic islands on the external kink mode stability [1]. On the other hand, unstable internal modes were revealed resonant to the $m = 1$ and $m = 2$ island chains (longitudinal wave number $n_h = 0$ in the helical coordinates). Higher order islands $m \geq 3$ do not lead to the instability.

Helical boundary deformation generating the islands at the resonant surface $q = m/n$ should possess the $2\pi/m$ rotational symmetry. That is why only the islands at $q = 2$ surface are unstable from all $q > 1$ islands. For $q = 1$ both $m = 1$ and $m = 2$ island chains are unstable against the $n_h = 0$ internal mode. The stability features of helical islands due to helical plasma deformation are very different from those of $n = 0$ axisymmetric islands that are formed in finite aspect ratio, reversed current density configurations [4], for which it was found that all island chains are unstable except for a $m = 4$ case [5].

The discovered instability of the equilibria with islands in helically symmetric large aspect ratio approximation may apply to the islands at $q = 1$ and $q = 2$ resonant surfaces in the tokamak plasma core, but the model validity is questionable for strongly shaped plasma cross sections.

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