Modulational instability and envelope structures in electron-positron-ion plasmas with warm ions

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Abstract

The nonlinear amplitude modulation and envelope solitons of ion acoustic wave are studied in the presence of warm ions in unmagnetized electron-positron-ion plasmas. The Krylov-Bogoliubov-Mitropolsky (KBM) method is used to derive the nonlinear Schrödinger equation. The dispersive and nonlinear coefficients are obtained which depends on the ion temperature and positron density in electron-positron-ion plasmas. The modulationally stable and unstable regions are studied numerically for a wide range of wave number. It is found that both ion temperature and positron density plays significant role in the formation of bright and dark envelope solitons in electron-positron-ion plasmas.

1. Set of Dynamic Equations

The basic set of normalized fluid equations for nonlinear propagation of ion acoustic waves with warm ions and isothermal Boltzmann distributed electrons and positrons are given as follows:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + \left(v\frac{\partial}{\partial x}\right)v = -\frac{\partial \Phi}{\partial x} - \frac{\sigma}{n}\frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial p}{\partial t} + \left(v\frac{\partial}{\partial x}\right)p = -\gamma p \frac{\partial v}{\partial x}, \tag{3}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = (1+\delta)\exp(\Phi) - \delta\exp(-\Phi) - n, \tag{4}$$

where electric field intensity is defined as $\boldsymbol{E} = -\nabla\phi$ and ϕ is the electrostatic potential. The normalized ion density and velocity are defined as $n = \frac{n_i}{n_{i0}}$ and $v = \frac{v_i}{c_s}$, respectively and ion acoustic speed is defined as $c_s = \sqrt{\frac{T_e}{m_i}}$. The normalization of time and space is done with

inverse of ion plasma frequency ω_{pi}^{-1} and effective Debye length λ_D respectively, which are

defined as $\omega_{pi} = \left(\frac{4\pi n_{i0} e^2}{m_i}\right)^{\frac{1}{2}}$ and $\lambda_D = \left(\frac{T_e}{4\pi n_{i0} e^2}\right)^{\frac{1}{2}}$. The normalized electrostatic potential and

ion pressure are defined as $\Phi = \frac{e\phi}{T_e}$ and $p = \frac{p_i}{p_{i0}}$ where $p_{i0} = n_{i0} T_i$ has been defined and T_i is

the ion temperature. It has also been assumed that $T_e = T_p$ and positron to ion density ratio

has been defined as $\delta = \frac{n_{p0}}{n_{i0}}$, so that $\delta = \frac{\beta}{(1-\beta)}$ and $\beta = \frac{n_{p0}}{n_{i0}}$ which lies in the range $0 \leq \beta < 1$,

while ion to electron temperature ratio is defined $\sigma = \frac{T_i}{T_e}$. The equilibrium $n_{e0} = n_{i0} + n_{p0}$ has

been defined. The equilibrium density of electrons and positrons are denoted as $n_{e0} (n_{p0})$,

while $T_e (T_p)$ is the electron (positron) temperature measured in the energy units. We have

assumed the same temperature of positrons and electrons because in ion dynamic scale both

fast moving species are generally taken to be in thermodynamic equilibrium.

2. Derivation of NLS Equation

In order to derive the NLS equation, Let S be the state (column) vector $(n, v, \phi, p)^T$ describing

the system's state at a given position x and instant t. We shall consider small deviations from

the equilibrium state $S^0 = (1,0,0,1)^T$ by assuming perturbation solution of the form

$$S = S^0 + \sum_{l=1}^{\infty} \epsilon^l S_l(a, \bar{a}, \psi). \tag{5}$$

In the above equation, $a, \bar{a},$ and ψ in the parentheses indicate that $n, v, \phi,$ and p depends

implicitly on x and t through a, $\bar{a}$ and ψ, where a is the complex amplitude, ψ is the carrier wave

phase defined as $\psi = (kx - \omega t)$, here k and ω are normalized wave vector and frequency,

respectively, and $\bar{a}$ is the complex conjugate of amplitude 'a'. The complex amplitude 'a' is

assumed to be slowly varying function of x and t such that it can be written as

$$\frac{\partial a}{\partial t} = \sum_{l=1}^{\infty} \epsilon^l A_l(a, \bar{a}), \quad \frac{\partial a}{\partial x} = \sum_{l=1}^{\infty} \epsilon^l B_l(a, \bar{a}) \tag{6}$$

along with the complex conjugate relation. The unknown functions A and B are to be

determined in such a way to make the solution (5) secularity free. Now substituting the above

equations into normalized set of equations and collecting the terms with the same power of ϵ.

If we choose the starting solution for $\Phi_1 = a\, e^{i\psi} + \bar{a}\, e^{-i\psi}$, then the first order solution leads

to the following dispersion relation or ion acoustic wave in the presence of warm ions in

unmagnetized e-p-i plasmas, i.e.,

$$\omega^2(1 + k^2 + 2\delta) - k^2 - \gamma k^2(1 + k^2 + 2\delta) = 0. \tag{7}$$

Now continuing the evaluation of the perturbation solution, we will find the second order solution $S_2(a, \bar{a}, \psi)$. The nonsecularity condition at $O(\epsilon^2)$ demands that the coefficient of $e^{\pm i\psi}$ terms be set equal to zero, which leads to the following condition, i.e. $A_1 + v_g B_1 = 0$, where $v_g = \frac{k}{\omega}\left[\frac{(1+2\delta)}{(1+k^2+2\delta)^2} + \gamma\sigma\right]$ represents the group velocity of the wave. After applying the secularity free condition, we obtain the second order solution for Φ_2 as follows,

$$\Phi_2 = G\, a^2\, e^{2i\psi} + b(a, \bar{a})e^{2i\psi} + c.c + c_1(a, \bar{a}), \qquad (8)$$

where $G = \frac{1}{6k^2}[3(1+k^2+2\delta)^2 - 1 + \gamma\sigma(1+\gamma)(1+k^2+2\delta)^3]$. Here $b(a, \bar{a})$ and its complex conjugate are constants with respect to ψ but functions of a and $\bar{a}$. In order to remove secularity, these constants should also be set equal to zero. The constant of integration $c_1(a, \bar{a})$ is assumed to be real. The complete set of second order secularity free solution $S_2(a, \bar{a}, \psi)$ comes out to be

$$
\begin{bmatrix} \phi_2 \\ n_2 \\ v_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} G \\ G_1 \\ G_2 \\ G_3 \end{bmatrix} a^2 e^{2i\psi} + \begin{bmatrix} 0 \\ H_1 \\ H_2 \\ H_3 \end{bmatrix} e^{2i\psi} + c.c + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} a\bar{a} + \begin{bmatrix} c_1(a, \bar{a}) \\ (1+2\delta)c_1(a, \bar{a}) \\ c_2(a, \bar{a}) \\ c_3(a, \bar{a}) \end{bmatrix} \qquad (9)
$$

The exact form of the coefficients G and H are given in the Ref.[1] and the constants of integration $c_1(a, \bar{a})$ and $c_2(a, \bar{a})$ are determined from $O(\epsilon^3)$ equation, which are given as follows:

$$c_1(a, \bar{a}) = \xi_1 a\bar{a} + R_1, \quad c_2(a, \bar{a}) = \xi_2 a\bar{a} + R_2, \quad c_3(a, \bar{a}) = \xi_3 a\bar{a} + R_3 \qquad (10)$$

here ξ_1, ξ_2 and ξ_3 are defined in Ref.[1].

On following the same procedure as for the second order solution, we can also find the third order solution $S_3(a, \bar{a}, \psi)$ from equations of $O(\epsilon^3)$. In order to have a solution S_3 to be secular free, we must set the coefficients of $e^{\pm i\psi}$ in the third order solution equal to zero, which would lead to the following equation:

$$i\left(A_2 + v_g B_2\right) + P\left(B_1\frac{\partial}{\partial a} + \bar{B}_1\frac{\partial}{\partial \bar{a}}\right)B_1 + Q|a|^2 a = Ra \qquad (11)$$

where coefficients P and Q are defined as

$$P = \frac{1}{2}\frac{dv_g}{dk} = -\frac{1}{2}\frac{\omega^5}{2k^4}\frac{[3(1+2\delta) + \gamma\sigma(1+k^2+2\delta)(3(1+2\delta) - k^2]}{[1+\gamma\sigma(1+k^2+2\delta)]^4}$$

$$Q = \frac{k^2}{2\omega(1+k^2+2\delta)^2}[GQ_1 + Q_2] + Q_3,$$

and Q_1, Q_2, Q_3 and R have been defined in Ref.[1].

Let us assume $t_2 = \epsilon^2 t$, $x_1 = \epsilon x$, and $x_2 = \epsilon^2 x$ and also by using expressions given in Eq.(6) one dimensional nonlinear Schrodinger equation (NLS) for ion acoustic wave with warm ions in unmagnetized e-p-i plasma is described as follows:

$$i\frac{\partial a}{\partial t} + P\frac{\partial^2 a}{\partial \zeta^2} + Q|a|^2 a = 0. \tag{12}$$

Here, dispersion coefficient P and the nonlinear interaction coefficient Q are above, respectively. We have introduced the following co-ordinate transformation i.e., ζ=ε(x-v$_g$t), τ=t$_2$= εt$_1$= ε^{2}t. Note that for simplicity, we have dropped the linear interaction term 'Ra' as it is not of much importance and simply causes a phase shift of the nonlinear structure. For cold ions and in the absence of positrons, we get the same expressions of coefficients P and Q as described in Ref. [2]. However, in the absence of positron case, we will get the same coefficients of P and Q as described in Refs. [3] for e-i plasma in the presence of warm ions.

Now we will analyze the modulational instability formation of envelope structures of the ion acoustic waves in the presence of warm ions in unmagnetized e-p-i plasmas as described by NLS equation (12). It is well known that the modulational instability depends on the sign of the product of the dispersive and nonlinearity coefficient, i.e., PQ. The ion acoustic wave trains in e-p-i plasma are modulationally stable or unstable depending on whether the product PQ is less or greater than zero, respectively. It is clear that dispersive and nonlinear coefficients depends on parameters such as positron density and ion temperature, which effects the stability criteria of the modulated ion acoustic wave over a wide range of wave number. It can be found easily that maximum growth rate of the modulated wave is given by $Q|a_0|^2$ at wave number $\left(\frac{K_c}{\sqrt{2}}\right)$, where the critical value of the wave number is given by $K_c = \sqrt{\frac{2Q}{P}} a_0$, and a_0 is the amplitude of the carrier wave. The solution of NLS equation is well documented which describes that the nonlinear excitations depend upon the sign of the product PQ i.e., the dispersive and nonlinear coefficients, respectively. These nonlinear excitations are either bright (self focusing region, PQ > 0) or dark (de-focusing region, PQ < 0) envelopes depending on the sign of the product PQ.

References

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