On the accuracy of the Debye shielding model

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The expression for the Debye shielding in plasma physics is usually derived under the assumptions that the plasma particles are weakly coupled, so their kinetic energy is much larger than the potential energy between them, and that the velocity distributions of the plasma species are Maxwellian. The first assumption also establishes that the plasma parameter N_D, the number of particles within a sphere with a Debye radius should be greater than λ_D, and determines the difference between weakly (such as ref. [8]) and strongly coupled plasmas. Under such assumptions, Poisson's equation can be linearised, and a simple analytic expression obtained for the electrostatic potential. However, textbooks (such as refs. [1-5]) rarely discuss the accuracy of this approximation. In this work we compare the linearised solution with the exact one, obtained numerically, and show that the linearisation, which underestimates the exact solution, is reasonably good even for $N_D \sim 40$. We give quantitative criteria to set the limit of the approximation when the number of particles is very small, or the distance to the test charge is too short.

In this work we compare the linearised solution of Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = -\frac{\rho_c(r)}{\varepsilon_o} \tag{1}$$

with the exact one, obtained numerically. We make the usual assumptions: (i) The kinetic energy of the particles is much greater than the potential energy between them, which means that the plasma is weakly coupled, and allows the linearization of the problem, (ii) Each particle species is in thermodynamic equilibrium, so their velocity distributions are Maxwellian, (iii) The positive ions are protons with infinite inertia, so they form a uniform background of density n. Since the electron density would then be given by $n_e(r) = n$

$exp(e\Phi(r)/kT)$, then $\rho_c(r) = n_e[\exp(-e\Phi(r)/kT) - \exp(e\Phi(r)/kT)]$. Also we are going to consider another way, writing $\rho'_c(r) = n_e[1 - \exp(e\Phi(r)/kT)]$ in order to compare with ref. [8].

Let us now define the Debye length $\lambda_D = (\varepsilon_0 kT/ne^2)^{1/2}$, and normalise the distance r and the potential $\Phi(r)$ in terms of it (equations 2 and 3). We have defined for convenience $N_D = 4\pi\lambda_D^3 n$, which is three times the number of particles within a sphere with a λ_D radius. Therefore, we can write the normalised distance and electrostatic potential

$$\rho = r / \lambda_D \quad , \tag{2}$$

$$\Psi(r) = 4\pi\varepsilon_o \lambda_D \Phi / e \quad . \tag{3}$$

Equation (1) can then be rewritten as equation (4) for ρ_c and (5) for ρ'_c.

$$\frac{d^2(\rho\Psi(\rho))}{d\rho^2} = -\rho N_D\left[exp\left(-\frac{\Psi(\rho)}{N_D}\right) - exp\left(\frac{\Psi(\rho)}{N_D}\right)\right] \quad , \tag{4}$$

$$\frac{d^2(\rho\Psi(\rho))}{d\rho^2} = -\rho N_D\left[1 - exp\left(\frac{\Psi(\rho)}{N_D}\right)\right] \quad . \tag{5}$$

Comparing the argument of the exponential in equations (4) and (5), we can see that the weakly coupled assumption $e\Phi(r)/ kT << 1$, which is a consequence of the potential energy between the particles being much smaller than their kinetic energy, is equivalent to the assumption that $N_D >> 1$. Under such conditions, we can keep the first two terms in the expansion of the exponential, and linearise equations (4) and (5) to yield the simple solution (6) for ρ_c, and (7) for ρ'_c.

$$\Psi_a(\rho) = \frac{exp(-2^{1/2}\rho)}{\rho} \tag{6}$$

$$\Psi_a(\rho) = \frac{exp(-\rho)}{\rho} \tag{7}$$

The main purpose is to compare the solution to equations (4) and (5), Ψ_e, obtained numerically, with the linearised equations (6) and (7), Ψ_a. The numerical solutions are obtained by means of a fourth-order Runge-Kutta routine, which is started at the tail of the solution, and integrated backwards. The initial values ρ_0, the distance ρ_d is such that the linearised and numerical solutions differ in *t*, where t is the tolerance $(|\Psi_a - \Psi_e| / \Psi_e) \times 100$.

Table 1. Initial value ρ_o and ρ_d for tolerances of *t* = 1%, 5%, and 10 %. For different values of N_D. ρ_o is the initial value for the Runge-Kutta calculation. ρ_d are the values at which the tolerances are reached.

				ρ_d					
		Step		*t* = 1%		*t* = 5%		*t* = 10%	
N_D	ρ_o	ρ_c	ρ'_c	ρ_c	ρ'_c	ρ_c	ρ'_c	ρ_c	ρ'_c
1	11	10^{-7}	10^{-8}	1.775	0.430	0.734	0.206	0.401	0.149
10	5	10^{-7}	10^{-8}	3.405×10^{-1}	1.098×10^{-1}	8.433×10^{-3}	7.346×10^{-3}	7.222×10^{-3}	6.814×10^{-3}
40	5	10^{-7}	10^{-8}	1.836×10^{-3}	1.585×10^{-3}	1.405×10^{-3}	1.380×10^{-3}	1.341×10^{-3}	1.326×10^{-3}
1×10^{2}	5	10^{-8}	10^{-9}	5.520×10^{-4}	5.369×10^{-4}	4.867×10^{-4}	4.838×10^{-4}	4.711×10^{-4}	4.692×10^{-4}
1×10^{3}	2	10^{-8}	10^{-9}	4.037×10^{-5}	4.026×10^{-5}	3.763×10^{-5}	3.760×10^{-5}	3.689×10^{-5}	3.688×10^{-5}
1×10^{4}	2	10^{-9}	10^{-10}	3.277×10^{-6}	3.274×10^{-6}	3.110×10^{-6}	3.110×10^{-6}	3.068×10^{-6}	3.068×10^{-6}
1×10^{5}	0.1	10^{-10}	10^{-11}	2.779×10^{-7}	2.777×10^{-7}	2.665×10^{-7}	2.665×10^{-7}	2.638×10^{-7}	2.639×10^{-7}
1×10^{6}	0.01	10^{-11}	10^{-12}	2.421×10^{-8}	2.419×10^{-8}	2.339×10^{-8}	2.339×10^{-8}	2.320×10^{-8}	2.321×10^{-8}
1×10^{7}	0.001	10^{-12}	10^{-13}	2.150×10^{-9}	2.148×10^{-9}	2.088×10^{-9}	2.088×10^{-9}	2.075×10^{-9}	2.075×10^{-9}

Table 1 shows, for several values of N_D, that the initial values ρ_0 used for the numerical integration, and the values ρ_d for which tolerances of 1, 5 and 10% between the two solutions fail. The same results are plotted in logarithmic scale in figure 1. For large values of N_D, the distance at which the approximation breaks is roughly the same regardless of the tolerance. In the plots in Fig. 1, they overlap down to N_D = 40. As N_D is further reduced, the distance at which the approximation breaks increases, and is larger for smaller tolerances.

Thus, the linear approximation is very good when N_D = 40 to 10^7, and it is possible to fit the results to a curve (solid lines in Fig. 1), which yields the empirical laws: $\rho_d(N_D^{1.06482}) = 0.0572822$ for ρ_c, and $\rho_d(N_D^{1.06489}) = 0.0573324$ for ρ'_c. This should be compared to the curve $\log_{10}\rho = -\log_{10}N_D$, which stands for the limit $N_D\rho = 1$. Therefore, our result gives a

quantitative meaning to the statement $N_D\rho << 1$. We should note that this result differs with that found in Ref. [8]: $N_D\rho_d = 0.01$. Our work goes further than that of Ref. [8], in that we study the cases of smaller values of N_D. Besides, only the case in which the tolerance is 10% was reported. The numerical (exact) solution for the electrostatic potential $\Psi(\rho)$ was obtained, from equation (5), and it was compared to the approximate solution (10), found for the linearised approximation. The latter underestimates the exact solution, but is a good approximation for values of N_D as small as 40. Even for $N_D = 5$, the approximation is still good, down to $r = 0.05\lambda_D$ if one requires only a 5% tolerance.

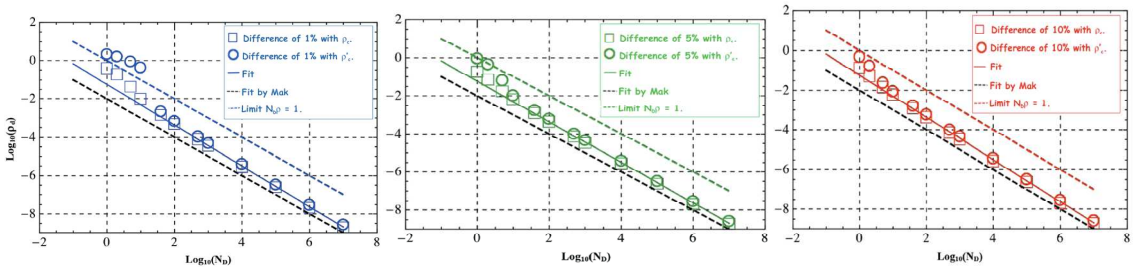

Figure 1. (left) t = 1%, (centre) t = 5%, (right) t = 10%.

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