Approach for efficient linear MHD analysis without spectrum pollution in tokamaks

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For realizing magnetic confinement fusion, it is necessary to suppress ideal magnetohydrodynamics (MHD) instabilities. Fortunately, linear ideal MHD theory has been well-established[1], and many numerical codes have been developed to identify linear ideal MHD stability with realistic plasma parameters under static conditions[2, 3, 4].

In the ideal case without plasma rotation, the theoretical background for non-polluted calculations was established[5]. Recently, however, based on the physics understanding achieved with theoretical and experimental analyses, rotation effects and non-ideal effects (e.g. effect from energetic particles, FLR effects) deemed to be important to accurately identify the stable operation regime. Since these rotation and/or non-ideal effects change the mathematical/numerical property of the problem to be solved from Hermitian to non-Hermitian, we have to treat the problem carefully from numerical viewpoints, and one of the promising solutions is the finite element method with higher-order element.

In this paper, we discuss a finite element method (FEM) which is applicable to identify the stability of linear ideal MHD modes without a spectrum pollution. As discussed in Ref.[5], a spectrum pollution observed in a spectrum analysis with FEM usually comes from the failure of physics constraint due to choosing in-appropriately the basis functions used for expressing independent vector variables. In the MHD spectrum code based on the energy principle[1], the important physics constraint is come from the resonant condition on rational surfaces and the plasma compressibility; details are discussed in Ref.[5]. In the cylindrical coordinate system, this constraint requires that the following conditions must be satisfied completely

$$\frac{d}{dr}(r\xi_r) + \imath m\left(\frac{B_z\xi_\theta - B_\theta\xi_z}{B_z}\right) = 0, \tag{1}$$

$$\imath\frac{r\xi_z}{B_z} = 0. \tag{2}$$

where ξ is the displacement in the cylindrical coordinate system (r,θ,z), and m is the poloidal Fourier mode number. The hybrid element method (HEM) can satisfy these conditions by choosing the basis function for X as e^p and those for V and Z as e^{p-1}, where p is the order of the basis function e.

In HEM, the conditions Eqs. (1) and (2) are satisfied everywhere in the system. This is the complete condition for avoiding a spectrum pollution, and is essential when the numerical integral of the system is obtained based on the Newton-Cotes formula with the non-conforming form finite element. On the other hand, when a quadrature rule, for example the Gauss-Legendre (G-L) quadrature, is used instead of the Newton-Cotes formula, this condition can be relaxed as that Eqs. (1) and (2) and need to be satisfied only at specified points within the domain of integration. The COOL method was developed to utilize this advantage[6], and has been applied for stability codes in plasmas[7, 8].

From the viewpoint of the physics constrains required for avoiding a spectrum pollution, the original HEM and the COOL method implement them successfully and realize to avoid that almost perfectly. However, to avoid completely a spectrum pollution, it is necessary to confirm the property of the numerical integration in the radial direction, and hence, we briefly revisit the finite element method used for identifying the ideal MHD stability based on the energy principle in an axi-symmetric system. For simplicity, we pay attention to the marginal ideal MHD stability, and hence, the plasma compressibility term and the plasma inertia term are neglected. In this case, the perturbed plasma potential energy expressed with the plasma displacement $\xi \propto exp(-\imath n\phi)$ can be written as

$$\begin{aligned} W_p &= \pi \int_0^a dr \int_0^{2\pi} d\theta \left[A \left| D_\theta(X) \right|^2 + B \left| \imath n V + \frac{1}{q} \frac{\partial (rX)}{\partial r} + hX + r\beta_{r,\theta} D_\theta(X) \right|^2 \right. \\ &\quad \left. + C \left| \frac{\partial (rX)}{\partial r} + \frac{\partial V}{\partial \theta} \right|^2 + E \left| X \right|^2 \right]. \end{aligned} \tag{3}$$

where $X = \xi \cdot \nabla r, V = r(\xi \cdot \nabla \theta - (\xi \cdot \nabla \phi)/q), D_\theta(X) = (1/q)\partial X/\partial \theta - \imath n X$, and other unknown values are defined in Ref.[4]. As mentioned before, the HEM and the COOL method use the order p basis function e^p for X and the order $p-1$ one e^{p-1} for V and Z. With these basis functions, a series of the degree of integrand (polynomial) can be obtained as $2p$, $2p-1$, and $2p-2$. This indicates that it is necessary to use the numerical integral method which can yield an exact result for polynomials of degree $2p$. However, the numerical integral method used in the original HEM with non-conforming form in Ref.[5] and the COOL method is that for polynomials of degree $2p-1$; in particular, since the current version of the COOL method satisfies the conditions Eqs. (1) and (2) and only at Gauss points, it is impossible to increase the degree of polynomial to be solved exactly.

To fix the problem, we apply the G-L quadrature to HEM. This is because HEM satisfies the conditions Eqs. (1) and (2) everywhere, and enables to choose arbitrary the points used for a numerical integral (Gauss points). For example, when e^p basis function is used for X, the

number of Gauss points, k, is determined as $k \geq p+1$, because the G-L quadrature with k can exactly estimate an result for polynomials of degree $2k-1$.

To confirm the importance of the order of numerical integral, the COOL method and HEM with the G-L quadrature (HEM-GL) are implemented into the the MINERVA stability code[9], and identify the edge localized MHD mode stability; MINERVA uses FEM only for the discretization in the radial direction, and the dependence on the poloidal angle is decomposed into a Fourier series. The shape and the profiles of the pressure p and the safety factor q are shown in Fig. 1, and this equilibrium rotates toroidally with the profile $\Omega_t/\omega_{A0} = 0.018(1-\psi^{48})^4$, where ω_{A0} is the toroidal Alfvén frequency at magnetic axis. Note that this equilibrium is unstable for $10 \leq n \leq 36$ ideal MHD modes.

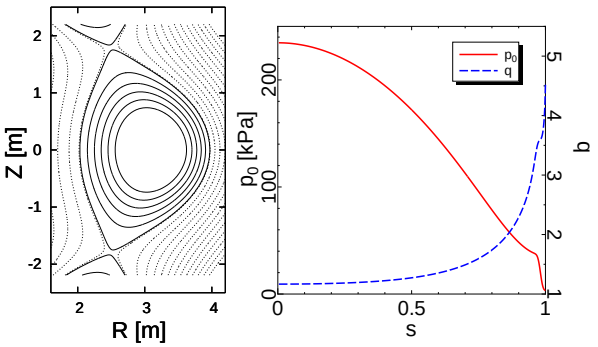

Figure 1: (a) Shape of the equilibrium. (b) Profiles of p and q.

We identify the stability of this equilibrium with the $p=2$ COOL method and the $p=2$ / $k=3$ HEM-GL, and confirm the numerical convergence of these methods on the radial mesh number NR; a variety of NR is 256, 512, 1024 and 1536. As shown in Fig. 2 (a), the $p=2$ COOL method requires more NR to identify the stability correctly, and in the case that NR is too few, un-physics results are observed. The radial structure of this un-physics results typically peaks near one of rational surface as shown in Fig. 2 (b); this is the $n=34$ un-physics result with $NR=512$, and this mode peaks near $nq=39$. On the other hand, with HEM-GL, no un-physics result is observed as shown in Fig. 3 (a), and the radial structure of the $n=34$ physics result can be obtained with $NR=512$ (Fig. 3 (b)); this is an edge-localized MHD mode with sheared toroidal rotation.

From these results, we have developed successfully a multi-order finite element method by applying the Gauss-Legendre quadrature numerical integral method into the hybrid element method. This method realizes to solve a non-Hermite linear ideal MHD equation including toroidal rotation without a spectrum pollution.

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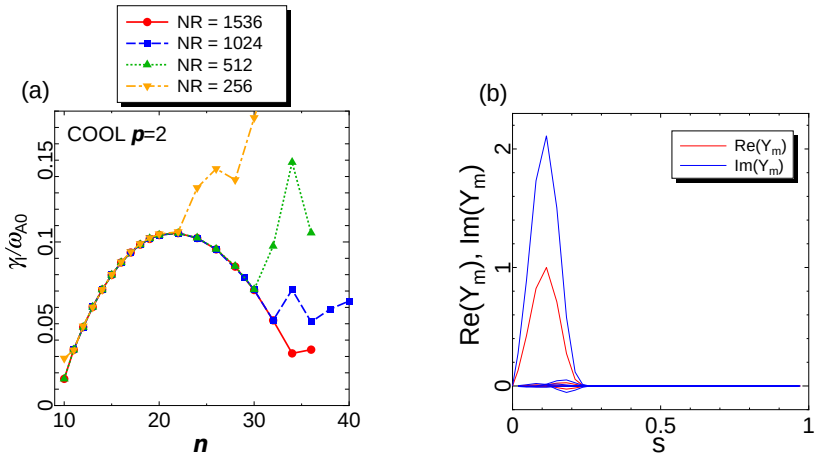

Figure 2: (a) Dependence of γ on n with different numbers of NR with the $p = 2$ COOL method. (b) Radial structure of $n = 34$ mode when $NR = 512$; $\gamma/\omega_{A0} = 0.1487$.

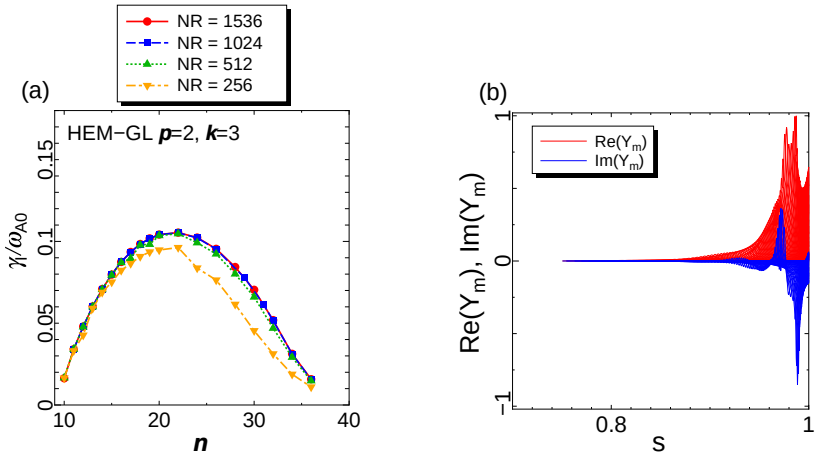

Figure 3: (a) Dependence of γ on n with different numbers of NR with the $p = 2$ / $k = 3$ HEM-GL. (b) Radial structure of $n = 34$ mode when $NR = 512$; $\gamma/\omega_{A0} = 0.0292$.

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