Self-organization of three-dimensional magnetohydrodynamics in confined geometries

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The spatiotemporal self-organization of visco-resistive magnetohydrodynamics (MHD) in a toroidal geometry is studied. It is observed that a flow is generated spontaneously, which evolves from dominantly poloidal to dominantly toroidal when the Lundquist number is increased. Up-down asymmetry of the geometry is necessary to generate a non-zero net toroidal mass-flow.

Shan and Montgomery [1] investigated the case of visco-resistive MHD in toroidal geometry. They showed that, in the case of uniform resistivity and imposed curl-free toroidal electric and magnetic fields, no static equilibria are possible. Since in the case they considered, the curl of the Lorentz-force is non-zero, the pressure gradient cannot balance the Lorentz-force, unlike the situation in cylindrical geometry. Therefore the magneto-fluid cannot remain quiescent, but will necessarily move. The possible steady state solutions in toroidal geometry with circular and non-circular cross-section were subsequently investigated in the work by Kamp and Montgomery [2].

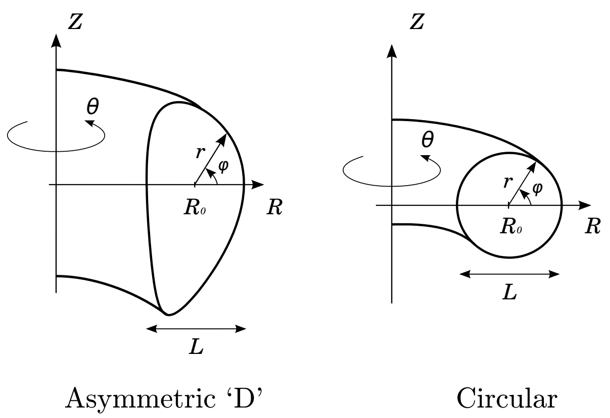

Figure 1: Cross-sections of the toroidal geometries considered in the present work. The toroidal direction is labelled θ and the poloidal φ.

At large Lundquist number, the dynamics of the visco-resistive MHD equations give rise to chaotic and turbulent behavior [3]. The resulting velocity field is then most probably not the one which is observed in the axi-symmetric time-independent case. In the present communication we therefore investigate the fully three-dimensional, non-stationary problem. The investigation of the dynamics of the full three-dimensional set of equations in complex geometry by direct numerical simulation at moderate Lundquist number has become possible only recently. We use the volume-penalization technique, an immersed boundary method, which we consider a good compromise between the ease of implementation, flexibility in geometry, and the numerical cost of the simulation. Results of confined visco-resistive MHD in two dimensions can be found in reference [4]. We recently extended this method to three-dimensions and in the present communication we present the results of simulations in toroidal geometries with circular and asymmetric-‘D’-shaped cross section.

MHD equations, dimensionless numbers and boundary conditions

We numerically solve the dimensionless incompressible viscoresistive MHD equations for the velocity field $\boldsymbol{u}$ and for the magnetic field $\boldsymbol{B}$, in ‘Alfvénic’ units [2],

$$\frac{\partial \boldsymbol{u}}{\partial t} - M^{-1}\nabla^2 \boldsymbol{u} = -\nabla\left(P + \frac{1}{2}\boldsymbol{u}^2\right) + \boldsymbol{u}\times\omega + \boldsymbol{j}\times\boldsymbol{B}, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - S^{-1}\nabla^2 \boldsymbol{B} = \nabla\times[\boldsymbol{u}\times\boldsymbol{B}], \tag{2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0, \tag{3}$$

with the current density $\boldsymbol{j} = \nabla\times\boldsymbol{B}$, the vorticity $\omega = \nabla\times\boldsymbol{u}$ and the pressure P. These equations are non-dimensionalized using the toroidal Alfvén speed $C_A = B_0/\sqrt{\rho\mu_0}$ as typical velocity, with B_0 the reference toroidal magnetic field at the center of the torus ($R = R_0$), ρ the density and μ_0 the magnetic constant. The reference length L (see Fig. 1) is the diameter of the cross section for the circular case and is the minor diameter for the asymmetric ‘D’ shape ($L = 0.94$ for both geometries). The dynamics are then governed by the initial and boundary conditions of the problem, and two dimensionless quantities: the viscous Lundquist number (M) and the Lundquist number (S) defined as

$$M = \frac{C_A L}{\nu}, \qquad S = \frac{C_A L}{\lambda}, \tag{4}$$

with λ the magnetic diffusivity and ν the kinematic viscosity. The ratio of these two quantities is the magnetic Prandtl number $P_r = \nu/\lambda$, which we have chosen unity in the present study, thereby reducing the number of free parameters, which characterize the magnetofluid, to one, the Lundquist number, M.

The initial condition for the simulation is zero velocity, and no-slip velocity boundary conditions are imposed. We consider the boundaries of the fluid domain as perfect conducting and coated with an infinitely thin layer of insulator. Thereby the normal component at the wall of the magnetic and current density field vanish. We impose curl-free toroidal magnetic and electric fields. Considering a uniform resistivity, this leads to the following relations for the toroidal magnetic field $\mathbf{B}_{\theta_{\mathbf{ext}}}$ and current density $\mathbf{J}_{\theta_{\mathbf{ext}}}$,

$$\mathbf{B}_{\theta_{\mathbf{ext}}}(R) = B_0\frac{R_0}{R}\boldsymbol{e}_\theta, \qquad \mathbf{J}_{\theta_{\mathbf{ext}}}(R) = J_0\frac{R_0}{R}\boldsymbol{e}_\theta. \tag{5}$$

The last equation is integrated numerically using the Biot-Savart law to determine the poloidal magnetic field $\mathbf{B}_{\varphi_{ext}}$. All the simulations presented in this communication are performed with $B_0 = 0.8$ and $J_0 = 0.3$. This corresponds, for both geometries, to a pinch ratio $\Theta \approx 0.16$, defined as the ratio between the wall-averaged poloidal and the volume-averaged toroidal magnetic field ($\Theta = \overline{B_\varphi} / < B_\theta >$). The only parameter that we vary is the Lundquist number M. The simulations are time-dependent and they are stopped when a dynamical steady state is reached.

Three-dimensional flows in toroidal geometries

The results in Fig. 2 show the presence of a poloidal flow, a pair of counter-rotating vortices in the $r - \varphi$ plane. These vortices move in opposite toroidal direction. This toroidal velocity increases with the Lundquist number M in the two considered geometries. For the circular cross section the three-dimensional velocity streamlines show a substantial change of topology from dominantly poloidal to dominantly toroidal flow (see Fig. 2).

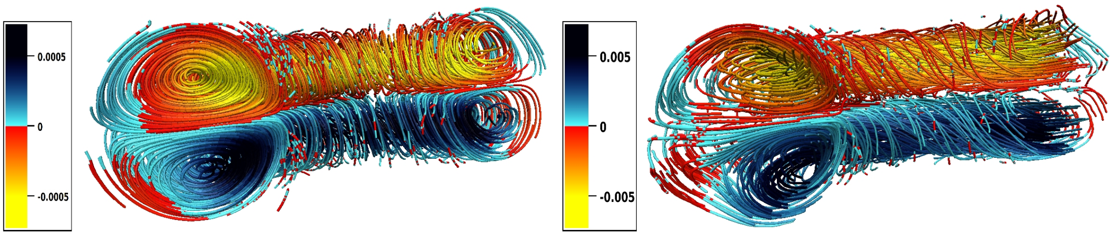

Figure 2: Streamlines colored with the value of the toroidal velocity, u_θ, for $M = 7.5$ (left) and $M = 75.2$ (right).

For both the symmetric and asymmetric geometry the square toroidal velocity saturates for increasing M at a value of $\sim 86\%$ of the total square speed (see Fig. 3). The principal direction of the flow motion is toroidal if M is raised beyond ~ 40.

For the asymmetric geometry, for low M, a poloidal pair of counter-rotating vortices appears as for the symmetric case. Similarly, if the Lundquist number is increased, an important toroidal flow develops. Unlike the symmetric case, there is a breaking of the symmetry in the flow and the part of the flow moving in the negative direction (the red and yellow zone), becomes larger on expense of the part of the flow which moves in the positive toroidal direction (blue zone). This symmetry breaking, illustrated in Fig. 5, leads to the development of a net toroidal flow. We quantify this using the volume-average toroidal angular momentum, $< L_\theta >= \frac{1}{V} \int_V R \cdot u_\theta dV$, which is non-zero (see Fig. 4). Its normalized value increases significantly with the Lundquist number.

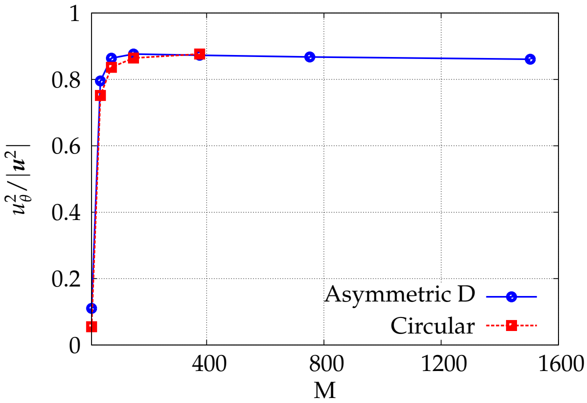

Figure 3: $u_\theta^2/|\boldsymbol{u}^2|$ as a function of M.

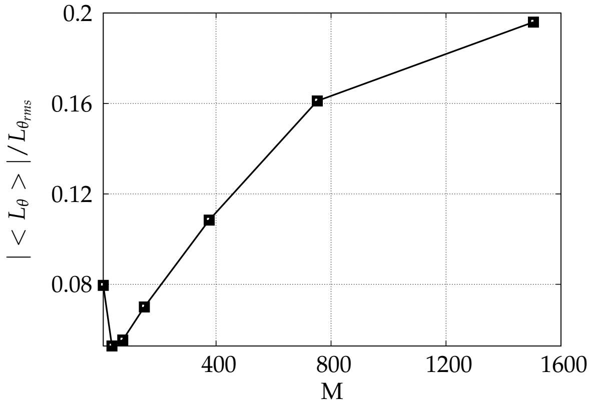

Figure 4: $|<L_\theta>|/L_{\theta_{rms}}$ as a function of M.

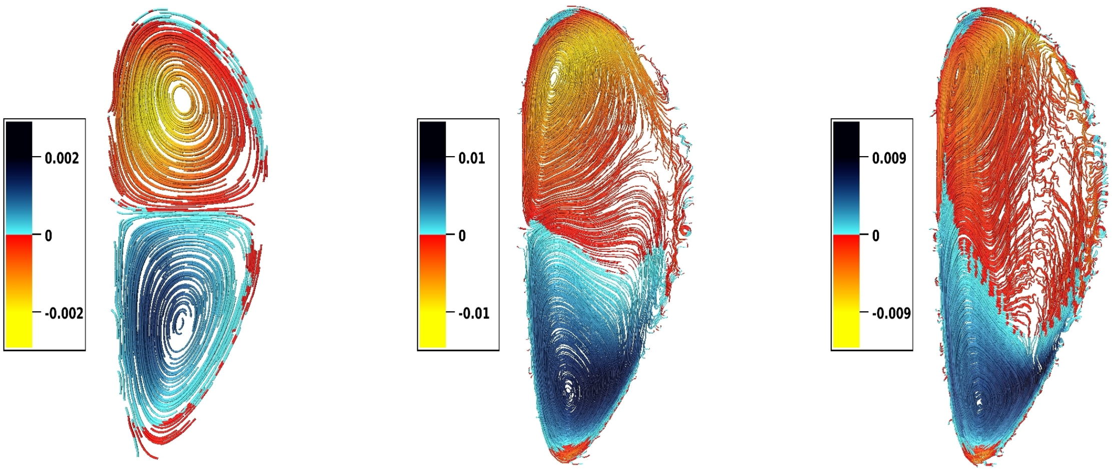

Figure 5: Streamlines (2D) colored with the value of the toroidal velocity, u_θ, for $M = 7.5$ (left), $M = 376$ (center) and $M = 1504$ (right).

We want to stress this last result: considering curl-free toroidal electric and magnetic fields and constant transport coefficients, visco-resistive magnetofluids spontaneously generate angular momentum, if the up-down symmetry of the torus is broken. This is a nonlinear effect which becomes negligible in the limit of small Lundquist number.

References

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