Resistive pressure driven RFP modes are not removed by heat conduction effects

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During the last decade it has been shown theoretically, numerically and experimentally that current driven, resistive *tearing* modes can be significantly suppressed in the reversed-field pinch (RFP). In these advanced scenarios, the confinement time can be enhanced by a factor 5-10. Pressure driven resistive instabilities (*g*-modes) still stand in the way, however, for high RFP confinement. Classical theory [1] shows that the unfavourable RFP curvature inevitably leads to unacceptably large linear growth rates even at high Lundquist numbers. Later theory [2] demonstrates, however, that the classical assumption of adiabatic plasma energy dynamics is inaccurate. The reason is that anomalously large experimental perpendicular heat conduction, together with strong parallel heat conduction, to a certain extent outbalance the pressure terms of the plasma energy equation. Resulting resistive length scales appear to extend the resistive layer at the resonance to allow for fully stable, finite beta RFP configurations. In the present work we show theoretically that the latter result is limited to low beta only and that it scales unfavourably with Lundquist number. Numerical solution, using a novel time-spectral method [3] of the linearised resistive MHD initial-value equations including heat conduction, ohmic heating and resistivity, supports the analytical results.

Resistive g-mode dispersion relation from classical adiabatic energy equation

In classical linear analysis of resistive instabilities in circularly cylindrical plasmas, the tearing layer near the resonance region plays a central role. Whereas ideal MHD is approximately valid outside this region, resistivity provides an important contribution in the resistive layer. Thermal conduction effects are, however, assumed to be negligible everywhere. Assuming perturbations $\propto \exp[i(m\theta + kz) + \gamma t]$, resonances occur at radial positions $r = r_s$ which satisfy $q = rB_{0z} / B_{0\theta} = -m / k$, in which relation the safety factor $q = q(r)$. Furthermore, $B_{0\theta} = B_{0\theta}(r)$ and $B_{0z} = B_{0z}(r)$ denote azimuthal and axial equilibrium magnetic fields. At the resonance radius, the helicity of the perturbation matches that of the equilibrium magnetic field to inhibit plasma stability through field line bending. By calculating the difference of the logarithmic derivative of the radial magnetic field perturbation outside and inside the resistive layer, using both the exterior ideal MHD model (with result denoted Δ') and the interior resistive MHD model, a dispersion relation, providing the growth rate for the perturbation with mode numbers (*m*,*k*), is obtained [1]:

$$L_r\Delta' = KQ^{5/4}(1-\frac{\pi D}{4Q^{3/2}}) \quad (1)$$

Here $K = 2\pi\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$, with Γ being the gamma function, and $L_r = S_0^{-1/3}(q/(kB_{0z}q'))^{1/3}$. Pressure effects enter through the normalized Suydam pressure gradient, for RFP equilibria approximately given by $D = -2p_0'(q/q')^2/(rB_{0z}^2)$ in which relation $p_0 = p_0(r)$ denotes equilibrium pressure. S_0 is the Lundquist number. Normalized growth rates γ are obtained from $Q = \gamma S_0^{1/3}(q/(kB_{0z}q'))^{2/3}$. Variables are normalized to plasma radius a, Alfvén time $\tau_A = a/v_A$, magnetic on-axis field B_{axis}, and plasma on-axis pressure $p_{axis} = B_{axis}^2/\mu_0 = \rho_{axis}v_A^2$ so that the normalized, uniform density and resistivity become $\rho = 1$ and $\eta = 1$, respectively. Dimensional resistivity is $\mu_0 a^2/(S_0\tau_A)$. All variables are evaluated at resonant point $r = r_s$.

The computation of Δ' is non-trivial for finite beta. Whereas for zero beta the expression for Δ' has a singularity of type x^{-1}, with $x = r - r_s$ being the distance from the singular layer, the singularity is of type x^{-2} for finite equilibrium pressure. Numerical procedures for calculation of Δ' for finite beta cases are given in [2,4]. In this analysis, a useful estimate of the resistive layer is $\delta = [\gamma r^2/(S_0(F')^2)]^{1/4}$, with $F = mB_{0\theta} + krB_{0z}$. Although the mode number *m* does not appear explicitly in Eq. (1) it is included in the computation of Δ', and the dispersion relation is valid for all (*m*,*k*).

It may be shown [1] that for confined equilibria where $D > 0$, Eq. (1) always predicts instability for the RFP, that is $\gamma > 0$. At low poloidal plasma beta (< 0.05), Eq. (1) features tearing mode scaling $\gamma \propto S_0^{-3/5}$ and at higher beta, resistive *g*-mode scaling $\gamma \propto S_0^{-1/3}$ emerges, being in agreement with numerical results using the resistive MHD model for the entire plasma region [5]. It should be noted that Eq. (5) of Ref [2] is different from Eq.(1) for finite beta and appears to be erroneous; it does not feature resistive *g*-mode scaling.

The dispersion relation (1) is approximative, being a result of several assumptions regarding orderings and limits. The most important are: 1) linearity and absence of mode coupling, 2) low beta, 3) small compressibility, 4) low resistivity, 5) neglect of ohmic heating and heat conduction terms in the energy equation, 6) the expansion $F = F'x$ and 7) perturbed variables vary on a short spatial scale, comparable to the resistive layer width.

Resistive g-mode dispersion relation including heat conduction

Including ohmic heating and heat conduction effects, the first order energy equation becomes

$$\frac{\partial p_1}{\partial t} = -v_{1r}p_0' - \Gamma p_0 \nabla\cdot\mathbf{v}_1 + \frac{\Gamma-1}{S_0}\{2(\eta_\perp j_{\perp 0} j_{\perp 1} + \eta_\parallel j_{\parallel 0} j_{\parallel 1}) + \frac{\chi_\parallel}{B_0^2}[-F^2 p_1/r^2 + iFp_0' B_{1r}/r] + $$
$$+\frac{\chi_\perp}{B_0^2}[F^2 p_1/r^2 - iFp_0' B_{1r}/r] + \chi_\perp[p_1'' + p_1'/r - (m^2/r^2 + k^2)p_1] + \chi_\perp' p_1'\} \qquad (2)$$

Here Γ denotes the ratio of specific heats. Braginskii [6] parallel heat conduction $\chi_\parallel$ may be used, but experimental perpendicular heat conduction $\chi_\perp$ exceeds that of Ref [6] by more than an order of magnitude. Thus it is preferably deduced from the equilibrium relation $(r\chi_\perp p_0')' = -r(\eta_\parallel j_{\parallel 0}^2 + \eta_\perp j_{\perp 0}^2)$. In Ref [2] a tearing order analysis, similar to that resulting in Eq. (1), is carried out assuming that Eq. (2) can be approximated by

$$0 = \frac{\chi_\parallel}{B_0^2}[-F^2 p_1/r^2 + iFp_0' B_{1r}/r] + \chi_\perp p_1'' \qquad (3)$$

The result is the dispersion relation (with all variables evaluated at $r = r_s$)

$$r_s\Delta' = KQ_B^{5/4} - \frac{\pi^{3/2} r_s D}{2\delta_x} \qquad (4)$$

The modified resistive layer scale length is $\delta_\chi = [(\chi_\perp/\chi_\parallel)(B_0^2/(m^2 B_{0\theta}^2))((rq)/q')^2]^{1/4}$. Growth rates γ are here obtained from $Q_B = \gamma S_0^{3/5}(r^2 q/(kB_{0z}q'))^{2/5}$. Again this result is only valid in the low beta limit, and rests on the same approximations as those listed for the classical dispersion relation. Also, the solution of Eq. (3) for p_1 depends on the questionable convergence of sums of Hermite polynomials.

The heat conduction modified dispersion relation (4) implies a possibility for resistive *g*-mode stable RFP equilibria [2]; any finite pressure equilibrium where $D > 0$ could avoid instability if Δ' is sufficiently negative. The equation is *delusive*, however. *First*, marginal stability ($Q = 0$) for any given equilibrium (fixed Δ') requires that the Suydam parameter $D \to 0$ as $S_0 \to \infty$. Thus, for reactor relevant values of plasma beta and Lundquist numbers, heat conduction does not provide essential stabilisation. Proof: in our normalisation, $\chi_\parallel \propto S_0^2$ and $\chi_\perp \propto S_0^0$, thus $\delta_\chi \propto S_0^{-1/2}$ and $\Delta' \propto -DS_0^{1/2}$. For fixed Δ', the marginal beta value thus tends to zero as $S_0 \to \infty$. An example: an equilibrium, stable below $\beta_p = 0.05$ for $S_0 = 10^4$, would be stable below $\beta_p = 0.0005$ for $S_0 = 10^8$. *Second*, equilibria marginally stable to tearing modes ($\Delta' = 0$) would obey the weak scaling $\gamma \propto S_0^{-1/5}$, practically a non-scaling which is not supported by neither resistive MHD simulations nor by experiments [7].

Fully resistive computations using GWRM code

To study the shortcomings of Δ' theory, a code for solving the complete set of linearised resistive MHD equations in the entire plasma domain has been developed. The full energy equation (2) is solved. The code is based on a new Generalized Weighted Residual Method (GWRM), where also the time domain is treated spectrally, thus avoiding the limitations of time stepping methods [3]. The table below summarizes some of the results. We use a typical RFP equilibrium, and use the standard ($\mu(r), p(r)$) equilibrium formulation so that marginal tearing mode stability at zero beta is guaranteed. We let $S_0 = 10^4$. Ohmic heating effects, which only slightly modify the results are left out, enabling comparison with the results of [2].

β_p	$\gamma_{adiabatic}$	γ_{Bruno}	γ_{GWRM}
0.05	$6.1\cdot10^{-3}$	$1.0\cdot10^{-3}$	stable
0.07	$7.7\cdot10^{-3}$	$1.9\cdot10^{-3}$	stable
0.10	$9.8\cdot10^{-3}$	$3.2\cdot10^{-3}$	$3.2\cdot10^{-4}$
0.12	$1.1\cdot10^{-2}$	$4.1\cdot10^{-3}$	$5.3\cdot10^{-4}$
0.15	$1.3\cdot10^{-2}$	$5.8\cdot10^{-3}$	$3.0\cdot10^{-3}$
0.20	$1.7\cdot10^{-2}$	$9.1\cdot10^{-3}$	$9.7\cdot10^{-3}$

The GWRM code verifies the stabilising effect of heat conduction on resistive *g*-modes at low beta, but classical growth rates are approached at higher beta, where Δ' results are invalid.

Conclusion

Pressure driven, resistive *g*-modes are ever-present in the RFP according to classical, adiabatic Δ' theory. More recent Δ' theory seems to predict that heat conduction effects cause substantial stabilisation due to extension of the resistive layer. It is here shown, analytically and numerically, that these effects are limited to low beta and low Lundquist numbers only. Consequently, reactor relevant stabilisation of pressure driven modes in the RFP must be sought among other physical effects.

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