Three-wave coupling coefficients for a magnetized plasma

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Abstract

The resonant interaction between three waves in a uniform magnetized plasma is reconsidered. Starting from previous kinetic expressions, that contain a general but too little used result, we are able to improve the formulas. This leads to an explicit expression for the three wave coupling coefficient which applies for arbitrary wave propagation in a magnetized Vlasov plasma.

Introduction

The resonant interaction between three waves in a plasma has now been studied during more than half a century. The first review of such processes in an unmagnetized plasma [1] was followed by specific studies for a magnetized plasma. In Ref. [2], a general theory for wave interactions in a cold one-component plasma was then developed. The numerous subsequent improvements of the theory were later reviewed in Refs. [3] and [4].

However, partly due to the lack of simple explicit expressions for the coupling coefficients, the production of alternative formulas continued. In the present paper we are going to discuss another expression which can be considered as the final result of previous efforts.

Results

Considering the resonant interaction between three waves with frequencies ω_j ($j = 1, 2, 3$) and wave vectors $\mathbf{k}_j$, we assume that the matching conditions

$$\omega_3 = \omega_1 + \omega_2 \tag{1}$$

and

$$\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \tag{2}$$

are satisfied. When calculating the coupling coefficients, it turns out that they contain a common factor V.

It is then possible to write the three coupled equations as

$$\frac{dW_{1,2}}{dt} = -2\omega_{1,2}\mathrm{Im}V \tag{3}$$

and

$$\frac{dW_3}{dt} = 2\omega_3\mathrm{Im}V \tag{4}$$

where $W = \varepsilon_0\mathbf{E}^* \cdot (1/\omega)\partial(\omega^2\boldsymbol{\varepsilon})\mathbf{E}$ is the wave energy, $\boldsymbol{\varepsilon}$ is the usual textbook dielectric tensor, and ImV stands for the imaginary part of V where ([3])

$$\mathrm{V} = \sum_s m \int d\mathbf{v}\mathbf{F}_0(\mathbf{v}) \sum_{\substack{p_1+p_2=p_3 \\ p_j=0,\pm1,\pm2,...}} I_1^{p_1}I_2^{p_2}I_3^{-p_3} \left[\frac{\mathbf{k}_1\cdot\mathbf{u}_{1p_1}}{\omega_{1d}}\mathbf{u}_{2p_2}\cdot\mathbf{u}_{3p_3}^* + \frac{\mathbf{k}_2\cdot\mathbf{u}_{2p_2}}{\omega_{2d}}\mathbf{u}_{1p_1}\cdot\mathbf{u}_{3p_3}^* + \frac{\mathbf{k}_3\cdot\mathbf{u}_{3p_3}^*}{\omega_{3d}}\mathbf{u}_{1p_1}\cdot\mathbf{u}_{2p_2} \right.$$

$$-\frac{i\omega_c}{\omega_{3d}}\left(\frac{k_{2z}}{\omega_{2d}}-\frac{k_{1z}}{\omega_{1d}}\right)\mathbf{u}_{3p_3}^*\cdot\left(\mathbf{u}_{1p_1}\times\mathbf{u}_{2p_2}\right)\Bigg] \tag{5}$$

where $\omega_{jd}=\omega_j-k_{jz}v_z-p_j\omega_c$, $I_j\,(=\exp(i\theta_j))=(k_{jx}+ik_{jy})/k_{j\perp}$, and the velocity $\mathbf{u}_{jp_j}$ satisfies

$$\omega_{jd}\mathbf{u}_{jp_j}+i\omega_c\hat{\mathbf{z}}\times\mathbf{u}_{jp_j}=\frac{iq}{m\omega_j}\left\{\omega_{jd}J_{p_j}\mathbf{E}_j+\left[\left(v_zE_{jz}+\frac{p_j\omega_c}{k_{j\perp}^2}\mathbf{k}_{j\perp}\cdot\mathbf{E}_{j\perp}\right)J_{p_j}+\right.\right.$$

$$\left.\left.\frac{iv_\perp\omega_c}{k_{j\perp}^2}(\hat{\mathbf{z}}\times\mathbf{k}_j)\cdot\mathbf{E}_j\frac{d}{dv_\perp}J_{p_j}\right]\mathbf{k}_j\right\} \tag{6}$$

where $J_{p_j}=J_{p_j}(k_{j\perp}v_\perp/\omega_c)$ denotes a Bessel function of order p_j.

The general theory for resonant three-wave interactions in plasmas then shows that the growth rate γ can be determined from [3]

$$\gamma^2=\frac{\omega_1\omega_2|\mathrm{V}|^2}{W_1W_2}. \tag{7}$$

Eq. (7) is very useful when rather complex kinetic effects are involved, for example in the interaction between two kinetic Alfvén waves and one ion-sound wave [5].

The development of, for example, the z-components (E_{jz}) of the wave electric field amplitudes is here governed by the three coupled bilinear equations (e.g. [3])

$$\frac{dE_{1z}^*}{dt}=\alpha_1E_{2z}E_{3z}^* \tag{8a}$$

$$\frac{dE_{2z}^*}{dt}=\alpha_2E_{1z}E_{3z}^* \tag{8b}$$

and

$$\frac{dE_{3z}}{dt}=\alpha_3E_{2z}E_{1z} \tag{8c}$$

where the z-axis is along the external magnetic field ($B_0\hat{\mathbf{z}}$), the star denotes complex conjugate, α_j are the coupling coefficients, $d/dt=\partial/\partial t+\mathbf{v}_{gj}\cdot\nabla+\nu_j$ where $\mathbf{v}_{gj}$ is the group velocity of wave j, and ν_j accounts for the linear damping rate. Formulas that determine the coefficients α_j for a magnetized Vlasov plasma have been derived previously [3], although in rather inexplicit forms. It is the purpose of the present paper to point out that *explicit* expressions for α_j can also be deduced from those formulas.

Starting from Eqs. (12)-(14) of Ref. [3] it turns out that we can write α_j in the comparatively simple form

$$\alpha_{1,2}=\frac{M_{1,2}}{\partial D(\omega_{1,2},\mathbf{k}_{1,2})/\partial\omega_{1,2}}C \tag{9a,b}$$

and

$$\alpha_3=-\frac{M_3}{\partial D(\omega_3,\mathbf{k}_3)/\partial\omega_3}C \tag{9c}$$

where the general coupling constant C as well as D and M_j are given by Ref. [6].

As a specific example we first mention the limiting case where all the waves are electrostatic. In that limit Eq. (6) reduces to

$$\mathbf{u}_{jp_j} = -\frac{iq}{m\omega_{jd}k_{zj}\left(1 - \omega_c^2/\omega_{jd}^2\right)}\left(\mathbf{k}_j - \frac{i\omega_c}{\omega_{jd}}\widehat{\mathbf{z}} \times \mathbf{k}_j - \frac{\omega_c^2}{\omega_{jd}^2}k_{zj}\widehat{\mathbf{z}}\right)J_{p_j} \tag{10}$$

whereas M_j in (9) is

$$M_j = \frac{k_{zj}^2 k_{\perp j}^2 c^4}{\omega_j^2} \tag{11}$$

and the dispersion function $D(\omega_j, \mathbf{k}_j)$ is described by the wellknown formula (c.f. Ref. [3])

$$D(\omega_j, \mathbf{k}_j) = \left(1 + \sum_{s,p}\frac{q^2}{m\varepsilon_0 k_j^2}\int \frac{d\mathbf{v}}{\omega_{jd}}\left(\frac{p\omega_c}{v_\perp}\frac{\partial F_0}{\partial v_\perp} + k_{zj}\frac{\partial F_0}{\partial v_z}\right)J_p^2\right)\frac{k_j^4 c^4}{\omega_j^4} \tag{12}$$

Thus the expressions for α_j are comparatively simple in the electrostatic limit.

Finally, we point out that Eqs. (10)-(12) have recently been generalized [6] so that arbitrary wave propagation is now also included. As the corresponding expressions for α_j are lengthy, we refer the reader to Ref. [6].

Conclusions

In the present paper we have improved the limiting results for three wave interactions in a cold plasma and pointed out that the *explicit* expressions for the coupling coefficients for wave interactions in a hot magnetized Vlasov plasma have been found recently. Our coupling coefficient C can thus be used as a starting point to estimate the coupling strength where the interaction between any kind of waves in a plasma has to be considered. It can also be useful in interpretations of stimulated scattering of electromagnetic waves in space plasmas. In the latter case we refer the reader to a short historical account of stimulated electromagnetic emissions in the ionosphere [7]. Our results can also play a key role in the theoretical interpretations of laser-fusion experiments.

References

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