# Analytic solution to the plasma equation with warm ions

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The general formulation of the problem of the plane-parallel symmetric discharge has been defined and solved with *ions born at rest* (cold ion-source model) to some extent, almost one century ago by Tonks and Langmuir [1], in so called two-scale approach (quasi-neutral plasma and electric field dominated plasma edge modelled separately from each other). The problem consists of simultaneous finding the electric field $E = -d\Phi/dx$ (or, equivalently, the potential $\Phi(x)$ ) and the ion velocity distribution function (VDF: $f_i(x,v)$) from Poisson and Boltzmann kinetic equations:

$$-\frac{d^2\Phi}{dx^2} = \frac{e}{\varepsilon_0}(n_i - n_e), \qquad v\frac{\partial f_i}{\partial x} - \frac{e}{m_i}\frac{d\Phi}{dx}\frac{\partial f_i}{\partial v} = S_i(x,v) \tag{1}$$

providing the ion source term $S_i(x,v)$ is specified and the electron density profile $n_e(\Phi)$ (usually Boltzmann-distributed) are known ($e$ - elementary charge, $\varepsilon_0$ - vacuum permeability, $m_{i,e}$ - ion and electron masses, $n_{i,e}$ - respective densities). Analytic solution to the problem under the cold-ion source scenario has been found under the assumptions that the potential profile is monotonic and that the motion of ions is fully determined by their total energy. The problem with ions created with initial velocities, however, is so demanding that a reliable numeric solution with a Maxwellian source with *arbitrary* ion temperature [2] has been obtained only recently [3], while the analytic solution hasn't been expected to be ever done. The first attempt into this direction, on contrary, appeared recently in Ref. [4] in the form of an *apparently explicit* expression:

$$\frac{-1}{E(\Phi)} = \frac{\sqrt{2\pi T_n}\,\mathrm{e}^{-(1+\frac{1}{2T_n})\Phi}}{\pi^2\sqrt{\Phi(\Phi_s-\Phi)}}\left[(1+\frac{1}{2T_n})\fint_{\Phi_s}^{0} dt\frac{\sqrt{t(\Phi_s-t)}}{t-\Phi}\,\mathrm{e}^{(1+\frac{1}{2T_n})t} - \frac{\pi\,\mathrm{e}^{(1+\frac{1}{2T_n})\frac{\Phi_s}{2}}}{\ln\left(\frac{16T_n}{\gamma_E|\Phi_s|}\right)}\mathrm{I}_0\left(\frac{(1+\frac{1}{2T_n})\Phi_s}{2}\right)\right], \tag{2}$$

$\Phi_s$ is the potential at the plasma-sheath boundary, to be either read from exact calculations of Kos et al. [3] or obtained from a newly found [4] algebraic implicit expression:

$$\left[1+\frac{2}{(1+\frac{1}{2T_n})\Phi_s\, ln\left(\frac{16T_n}{\gamma_E|\Phi_s|}\right)}\right] I_0\left(\frac{(1+\frac{1}{2T_n})\Phi_s}{2}\right) = I_1\left(\frac{(1+\frac{1}{2T_n})|\Phi_s|}{2}\right), \tag{3}$$

where $I_0(z)$ and $I_1(z)$ are familiar Bessel functions. Above expressions apply to the case of the Maxwellian ion source proportional to the local electron density, while extension to other physical scenarios are referred to in Ref. [5]. The temperatures, potential and energy here are normalised to the electron temperature $T_{e0}$ in the centre of the discharge, velocities to the so-called "ion-sound" $c_{s,0} \equiv \sqrt{kT_{e0}/m_i}$ multiplied by $\sqrt{2}$ [3], the coordinate $x$ to the characteristic system length [5] and with other quantities emerging accordingly, while $\gamma_E = \exp(C_E) = 1.78107$ (with the Euler-Macsheroni constant $C_E = 0.57721...$). The main parameter of the problem is the normalised neutral source temperature $T_n$, while the ion temperature turns out to be a local quantity [3] to be found from local ion VDF. Unfortunately, in addition to fact that Eq. 3 is not quite user-friendly, it holds in the limit for rather high ion-source temperatures, as illustrated in Fig. 1 where we represent sheath edge potential $\Phi_s$ as obtained from Eq. 3, in compar-

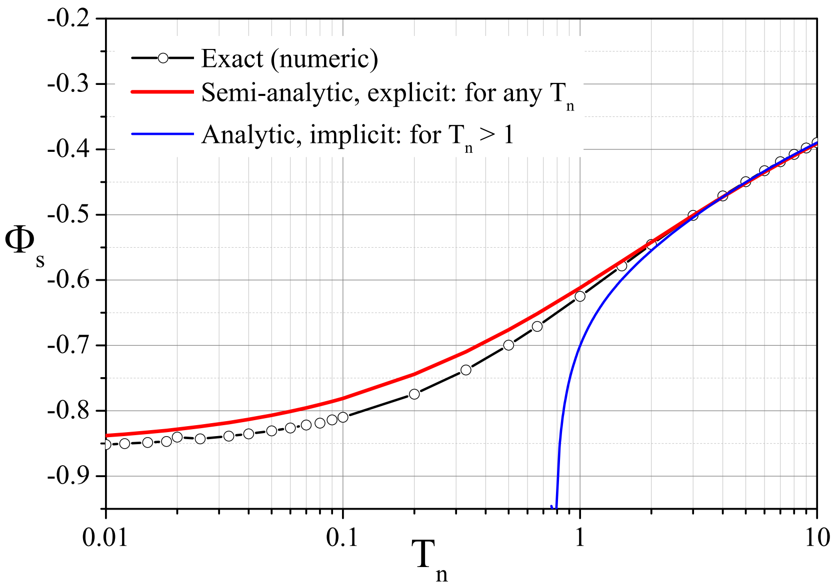


Figure 1: Exact plasma edge potential in comparisons with present explicit semi-analytic and our "old" implicit approximation

ison with the exact results from refined sheath-edge potential profile investigations [6]. Therefore, instead of it in this work we use our trial expression:

$$\Phi_s(T_n) \approx -\frac{1}{\ln\left(1.9T_n^{\frac{\sqrt{2}}{2}} + \exp\left[\frac{1}{0.854}\right]\right)} \tag{4}$$

which fits exact result much better than Eq. 3 as obvious from Fig. 1. In addition to the fact that our old "explicit" formula 2 is rather unreliable for $T_n \leq 3$ and completely fails slightly bellow $T_n = 1$, it contains *not-tabulated* integrals, so it is not very convenient for practical applications. Therefore, in present work we apply the two-point quasi-fractional approximation (see e.g., Ref. [7]) and use the properties of the electric field near $\Phi = 0$, and $\Phi = \Phi_s$ for constructing our basic result of the present work:

$$E \approx -\frac{40}{27\sqrt{T_n}} \frac{\sqrt{-\Phi}}{\sqrt{\Phi - \Phi_s}} \tag{5}$$

which fits excellently result obtained from Ref. 3 and holds as long as referred to approximations near $\Phi = 0$, and $\Phi = \Phi_s$ hold (i.e., according to Ref [6], for $T_n \simeq 0.06 < \infty$). In Fig. 2 we show the electric fields obtained with formula 5 in comparison with exact numerical method for various ion temperatures (note the differences between the ion source temperature $T_n$ and resulting ion temperature $T_i$ [3] taken in this particular example *at the sheath edge*. The error for $T_i = 0.42$ ($T_n = 1$) is due to usage of approximate formula 4, while alternative use of tabulated exact $\Phi_s$ , *removes this discrepancy completely* (not shown here). Once $E(\Phi)$ is found, the ion VDF can be calculated in a straightforward

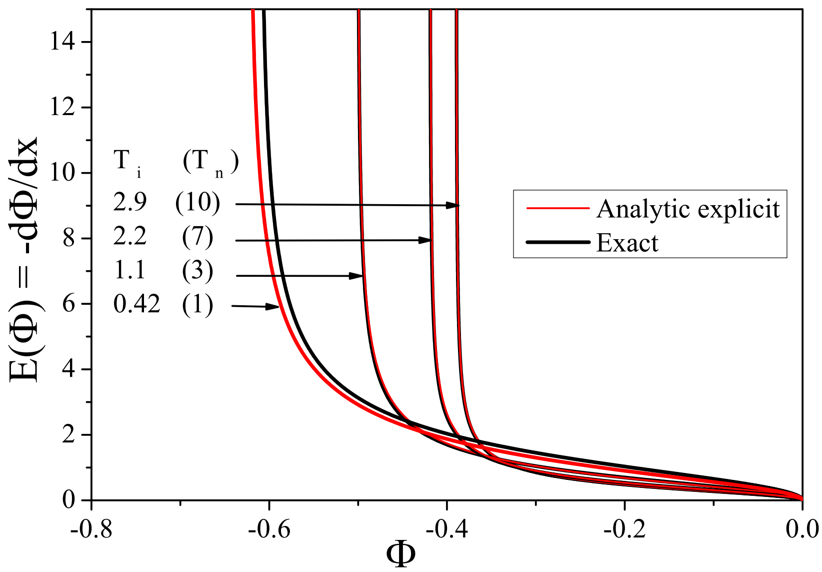


Figure 2: Approximate electric field in comparison with exact results

manner from:

$$f_i(\Phi(x), v) = \frac{1}{\sqrt{2\pi T_n}} \int_{\Phi'} \Psi(\Phi') \exp[\beta\Phi'] \exp[(\Phi' - \Phi)/T_n] d\Phi' \times \frac{\exp(-v^2/T_n)}{\sqrt{v^2 - (\Phi' - \Phi)}}. \tag{6}$$

taking into account the singularity of the kernel $E(\Phi)$, i.e., in the limits $(\Phi_s, \Phi_s + v^2/2)$ and $(\Phi_s, 0)$ for the branch left and right from $\sqrt{\Phi_s}$, respectively. Corresponding fluid quantities, i.e., moments of ion VDF as fiunctions of the local potential can be then obtained in a straightforward manner. In Fig. 3 we show the in VDFs at the plasma

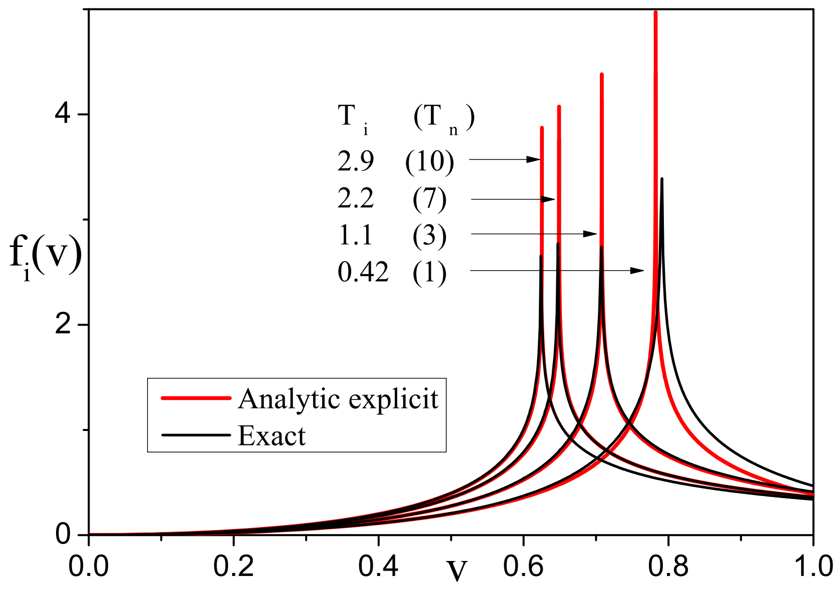


Figure 3: Ion velocity distributions at the plasma edge for various ion-temperatures

edge for illustrating the results which correspond to kernels from Fig. 2. The illustrated "error" due to use of Eq. 4 can be readily removed via applying exact tabulated $\Phi_s$ instead. Anyway, the shape of the velocity distribution, which is now trivial to obtain and make derivatives of it with unlimited precision is of extreme importance for further work on the plasma sheath problem, in particular for the plasma-sheath matching and resolving the controversies about the discrepancies between the kinetic and fluid Bohm criteria standing for more than 60 years opened.

**Acknowledgements**

This work was supported by the European Commission under (i) the Contract of Association between EURATOM and the Austrian Academy of Sciences and by (ii) the Austrian Science Fund (FWF) under projects P19333-N16, P9133 and P22345-N16. It was carried out within the framework of the European Fusion Development Agreement. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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