Exact conservation laws for truncated gyrokinetic Vlasov-Poisson equations

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Exact conservation laws for the gyrokinetic Vlasov-Poisson equations can either be derived from a variational principle by the Noether method [1] or directly if exact invariants for gyrocenter Hamiltonian dynamics are known [2]. We begin our Noether derivation with the non-canonical gyrocenter phase-space Lagrangian

$$\Gamma_{\rm gy} \;\equiv\; \left[\left(\frac{e}{c}\mathbf{A} \,+\, p_{\|}\,\widehat{\mathsf{b}}\right)\cdot \mathsf{d}\mathbf{X} \,-\, W\,\mathsf{d}t\right] \,-\, \left(H_{\rm gy} - W\right)\,\mathsf{d}\tau, \tag{1}$$

where W is the extended phase-space energy coordinate and the gyrocenter Hamiltonian is

$$H_{\rm gy}(\mathbf{X}, p_{\|}, \mu, t;\, \Phi_1) \;\equiv\; \mu\, B + \frac{p_{\|}^2}{2m} \,+\, \varepsilon\, e\,\langle \Phi_{1\rm gc}\rangle. \tag{2}$$

The gyrocenter Euler-Lagrange equation associated with an arbitrary displacement $\delta\mathbf{X}$ is

$$\frac{e}{c}\,\frac{d_{\rm gy}\mathbf{X}}{dt}\times\mathbf{B}^* \,-\, \frac{d_{\rm gy}p_{\|}}{dt}\,\widehat{\mathsf{b}} \,-\, \nabla H_{\rm gy} \;=\; 0, \tag{3}$$

where $\mathbf{B}^* \equiv \mathbf{B} + (p_{\|}c/e)\,\nabla\times\widehat{\mathsf{b}}$ and the gyrocenter canonical momentum is $\mathbf{p}_{\rm gy} \equiv (e/c)\,\mathbf{A} + p_{\|}\,\widehat{\mathsf{b}} \equiv (e/c)\,\mathbf{A}^*$, from which we obtain the Hamilton equation for the canonical momentum [1]

$$\frac{d_{\rm gy}\mathbf{p}_{\rm gy}}{dt} \;=\; -\,\nabla H_{\rm gy} \,+\, \frac{e}{c}\,\nabla\mathbf{A}^*\cdot\frac{d_{\rm gy}\mathbf{X}}{dt}. \tag{4}$$

In axisymmetric tokamak geometry, the magnetic vector potential is $\mathbf{A} \equiv -\,\psi\,\nabla\varphi + \Psi(\psi)\,\nabla\vartheta$, so that the magnetic field $\mathbf{B} \equiv \nabla\varphi\times\nabla\psi + q(\psi)\,\nabla\psi\times\nabla\vartheta$ yields the identity $\nabla\psi \equiv \mathbf{B}\times\partial\mathbf{X}/\partial\varphi$.

We now derive the Hamilton equation for the toroidal canonical gyrocenter momentum

$$p_{{\rm gy}\varphi} \;\equiv\; \frac{\partial\mathbf{X}}{\partial\varphi}\cdot\mathbf{p}_{\rm gy} \;=\; -\,\frac{e}{c}\,\psi \,+\, p_{\|}\,b_\varphi, \tag{5}$$

where $b_\varphi \equiv \widehat{\mathsf{b}}\cdot\partial\mathbf{x}/\partial\varphi$ denotes the covariant toroidal component of the magnetic unit vector. By taking the toroidal projection of the gyrocenter Euler-Lagrange equation (3), we obtain [1] $d_{\rm gy}p_{{\rm gy}\varphi}/dt \equiv -\,\partial H_{\rm gy}/\partial\varphi$, where we used the identity

$$\frac{\partial\mathbf{C}}{\partial\varphi} \,+\, \nabla\left(\frac{\partial\mathbf{x}}{\partial\varphi}\right)\cdot\mathbf{C} \;=\; \frac{\partial\mathbf{C}}{\partial\varphi} \,+\, \mathbf{C}\times\widehat{\mathsf{z}} \;\equiv\; 0, \tag{6}$$

which is valid for an arbitrary vector field $\mathbf{C}$ in axisymmetric tokamak geometry.

The truncated gyrokinetic Vlasov-Poisson equations are derived from the action functional

$$\mathscr{A}_{\rm gy} = \int_{x^4}\left(\frac{\varepsilon^2\,|\mathbf{E}_1|^2}{8\pi} - \frac{|\mathbf{B}|^2}{8\pi}\right) + \frac{\varepsilon^2}{2}\int_{z^7} F_0\, e\left\langle \pounds_1\,\Phi_{\rm 1gc}\right\rangle - \int_{Z^8} \mathscr{F}_{\rm gy}(Z)\,\mathscr{H}_{\rm gy}(Z;\Phi_1), \quad (7)$$

where summation over particle species is implied, with the extended gyrocenter Hamiltonian $\mathscr{H}_{\rm gy} \equiv H_{\rm gy} - W$ and the extended gyrocenter Vlasov distribution $\mathscr{F}_{\rm gy} \equiv c\,\delta(W - H_{\rm gy})\,F$. In (7), the gyrocenter Lie derivative is $\pounds_1\Phi_{\rm 1gc} \equiv (e/\Omega)\,\{\widetilde{\Phi}_{\rm 1gc},\,\Phi_{\rm 1gc}\}_{\rm gc}$, where $\widetilde{\Phi}_{\rm 1gc} \equiv \int \widetilde{\phi}_{\rm 1gc}\,d\theta$ and $\{\;,\;\}_{\rm gc}$ is the guiding-center Poisson bracket. The gyrokinetic variational principle $\delta\mathscr{A}_{\rm gy} \equiv \int\,\delta\mathscr{L}_{\rm gy}\,d^4x = 0$ introduces the variation of the gyrokinetic Lagrangian density

$$\delta\mathscr{L}_{\rm gy} = -\varepsilon\,\delta\Phi_1\left[e\int_{z^6}\left(\langle\delta^3_{\rm gc}\rangle\,F - \varepsilon\,F_0\left\langle\pounds_1\,\delta^3_{\rm gc}\right\rangle\right)\right] + \frac{\varepsilon^2}{4\pi}\,(\delta\mathbf{E}_1\cdot\mathbf{E}_1) - \int\,\delta\mathscr{F}_{\rm gy}\,\mathscr{H}_{\rm gy}\,d^4p, \quad (8)$$

where $\delta\mathscr{F}_{\rm gy} \equiv \{\mathscr{S}_{\rm gy},\,\mathscr{F}_{\rm gy}\}_{\rm gc}$ is generated by the canonical generating function $\mathscr{S}_{\rm gy}$ and $\delta\mathbf{E}_1 \equiv -\nabla\delta\Phi_1$. From the variational principle, we obtain the gyrokinetic extended Vlasov equation $\{\mathscr{F}_{\rm gy},\,\mathscr{H}_{\rm gy}\}_{\rm gc} = 0$, which, when integrated over W, yields the gyrocenter Vlasov equation [1]

$$\frac{\partial F}{\partial t} + \frac{d_{\rm gy}\mathbf{X}}{dt}\cdot\nabla F + \frac{d_{\rm gy}p_\|}{dt}\,\frac{\partial F}{\partial p_\|} = 0. \quad (9)$$

We also obtain the gyrokinetic Poisson equation

$$\frac{\varepsilon\,\nabla\cdot\mathbf{E}_1}{4\pi} = e\int_{z^6}\left(F\langle\delta^3_{\rm gc}\rangle - \varepsilon\,F_0\,\left\langle\pounds_1\,\delta^3_{\rm gc}\right\rangle\right) \equiv \rho - \nabla\cdot\mathbb{P}, \quad (10)$$

where ρ denotes the gyrocenter charge density and the gyrokinetic polarization

$$\mathbb{P} = \frac{\widehat{\mathsf{b}}}{\Omega}\times\left[\int F\left(e\,\frac{d_{\rm gy}\mathbf{X}}{dt}\right)d^3p\right] \equiv \mathbb{P}_{\rm gc} + \mathbb{P}_{\rm gy} \quad (11)$$

includes contributions from the guiding-center polarization (from the guiding-center velocity $d_{\rm gc}\mathbf{X}/dt$) and the gyrocenter polatization (from the perturbed $E\times B$ velocity $\varepsilon\,(c\widehat{\mathsf{b}}/B)\times\nabla\langle\Phi_{\rm 1gc}\rangle$).

By inserting (9)-(10) into (8), we obtain the gyrokinetic Noether equation $\delta\mathscr{L}_{\rm gy} \equiv \partial\Lambda/\partial t + \nabla\cdot\Gamma$, where the Noether fields are

$$\Lambda \equiv \int\,\mathscr{S}_{\rm gy}\,\mathscr{F}_{\rm gy}\,d^4p \;\;\text{and}\;\; \Gamma \equiv -\,\frac{\varepsilon^2\,\delta\Phi_1}{4\pi}\,\mathbf{E}_1 + \int\left(\mathscr{S}_{\rm gy}\,\mathscr{F}_{\rm gy}\right)\frac{d_{\rm gy}\mathbf{X}}{dt}\,d^4p. \quad (12)$$

The gyrokinetic Noether equation is now used to derive the gyrokinetic toroidal angular-momentum conservation law. First, when considering arbitrary infinitesimal displacements $\delta\mathbf{x}$, we obtain the Noether momentum equation [1]

$$\frac{\partial\mathbf{P}}{\partial t} + \nabla\cdot\Pi_{\rm gy} = -\int F\left(\nabla H_{\rm gy} - \frac{e}{c}\,\nabla\mathbf{A}^*\cdot\frac{d_{\rm gy}\mathbf{X}}{dt}\right), \quad (13)$$

where

$$\mathbf{P} = \int F\,\mathbf{p}_{\rm gy}\,d^3p \;\;\text{and}\;\; \Pi_{\rm gy} = \int F\,\frac{d_{\rm gy}\mathbf{X}}{dt}\,\mathbf{p}_{\rm gy}\,d^3p. \quad (14)$$

We note that (13) can also be obtained as the gyrocenter-Vlasov moment of (4). Second, we consider the infinitesimal toroidal rotation $\delta\mathbf{x} \equiv \delta\varphi\,\partial\mathbf{x}/\partial\varphi = \delta\varphi\,\widehat{\mathsf{z}}\times\mathbf{x}$. The toroidal projection of (13) yields the gyrokinetic toroidal angular-momentum equation [1, 2]

$$\frac{\partial P_\varphi}{\partial t} + \nabla\cdot\mathbf{Q}_\varphi = -\,\varepsilon\,e\int F\,\frac{\partial\langle\Phi_{1\mathrm{gc}}\rangle}{\partial\varphi}\,d^3p, \tag{15}$$

where

$$P_\varphi \equiv \mathbf{P}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} = \int F\,p_{\mathrm{gy}\varphi}\,d^3p \;\;\text{and}\;\; \mathbf{Q}_\varphi \equiv \Pi_{\mathrm{gy}}\cdot\frac{\partial\mathbf{x}}{\partial\varphi} = \int F\,\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\,p_{\mathrm{gy}\varphi}\,d^3p,$$

and we used the identity (6) to obtain

$$(\nabla\cdot\Pi_{\mathrm{gy}})\cdot\frac{\partial\mathbf{x}}{\partial\varphi} = \nabla\cdot\left(\Pi_{\mathrm{gy}}\cdot\frac{\partial\mathbf{x}}{\partial\varphi}\right) - \Pi_{\mathrm{gy}}^\top:\nabla\left(\frac{\partial\mathbf{x}}{\partial\varphi}\right) = \nabla\cdot\mathbf{Q}_\varphi + \int F\left(\frac{e}{c}\frac{\partial\mathbf{A}^*}{\partial\varphi}\cdot\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\right).$$

Third, we introduce the operation of magnetic-surface average $[\![\cdots]\!] \equiv \mathcal{V}^{-1}\oint(\cdots)\,\mathcal{J}\,d\vartheta\,d\varphi$, with the magnetic–coordinate (ψ,θ,φ) Jacobian $\mathcal{J} \equiv (\nabla\psi\times\nabla\theta\cdot\nabla\varphi)^{-1}$ and $\mathcal{V} \equiv \oint\mathcal{J}\,d\vartheta\,d\varphi$. Next, we introduce the gyrokinetic parallel-toroidal momentum

$$P_{\|\varphi} \equiv P_\varphi + \frac{\psi}{c}\,\rho = \left(\int F\,p_\|\,d^3p\right)b_\varphi, \tag{16}$$

and obtain the surface-averaged gyrokinetic parallel-toroidal momentum equation

$$\frac{\partial[\![P_{\|\varphi}]\!]}{\partial t} = -\,\frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\,[\![Q_\varphi^\psi]\!]\right) + \frac{\psi}{c}\frac{\partial[\![\rho]\!]}{\partial t} - \varepsilon\,e\left[\!\!\left[\int F\,\frac{\partial\langle\Phi_{1\mathrm{gc}}\rangle}{\partial\varphi}\,d^3p\right]\!\!\right], \tag{17}$$

where

$$[\![Q_\varphi^\psi]\!] = \left[\!\!\left[\int F\,\frac{d_{\mathrm{gy}}\psi}{dt}\,p_{\mathrm{gy}\varphi}\,d^3p\right]\!\!\right] \equiv [\![Q_{\|\varphi}^\psi]\!] - \frac{\psi}{c}\,[\![\nabla\psi\cdot\mathbf{J}]\!],$$

and $[\![\nabla\psi\cdot\mathbf{J}]\!] \equiv e\,[\![\int F\,(d_{\mathrm{gy}}\psi/dt)\,d^3p]\!]$. Lastly, we use the gyrocenter charge conservation law

$$\frac{\partial[\![\rho]\!]}{\partial t} = -\,[\![\nabla\cdot\mathbf{J}]\!] \equiv -\frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\,[\![\nabla\psi\cdot\mathbf{J}]\!]\right), \tag{18}$$

so that (17) becomes

$$\frac{\partial[\![P_{\|\varphi}]\!]}{\partial t} = -\,\frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\,\left[\!\!\left[Q_{\|\varphi}^\psi\right]\!\!\right]\right) + e\left[\!\!\left[\int F\left(\frac{1}{c}\frac{d_{\mathrm{gy}}\psi}{dt} - \varepsilon\frac{\partial\langle\Phi_{1\mathrm{gc}}\rangle}{\partial\varphi}\right)d^3p\right]\!\!\right] \tag{19}$$

where $c^{-1}\,d_{\mathrm{gy}}\psi/dt - \varepsilon\,\partial\langle\Phi_{1\mathrm{gc}}\rangle/\partial\varphi$ represents the gyrocenter toroidal electric field. If we now use the gyrokinetic quasineutrality condition $\rho \equiv \nabla\cdot\mathbb{P}$, the gyrocenter charge conservation law (18) becomes (with $\mathscr{P}^\psi \equiv \nabla\psi\cdot\mathbb{P}$)

$$\frac{\partial[\![\rho]\!]}{\partial t} \equiv \left[\!\!\left[\nabla\cdot\frac{\partial\mathbb{P}}{\partial t}\right]\!\!\right] = \frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\,\frac{\partial[\![\mathscr{P}^\psi]\!]}{\partial t}\right)$$

which implies the gyrocenter ambipolarity condition $[\![J^{\psi}_{\rm phys}]\!] \equiv [\![(\partial \mathscr{P}^{\psi}/\partial t) + \nabla\psi\cdot\mathbf{J}]\!] \equiv 0$, where the magnetization-current contribution vanishes since $[\![\nabla\psi\cdot\nabla\times\mathbf{M}]\!] = [\![\nabla\cdot(\mathbf{M}\times\nabla\psi)]\!] \equiv 0$. The parallel-toroidal momentum equation (19) thus becomes

$$\frac{\partial}{\partial t}\left([\![P_{\|\varphi}]\!] + \frac{1}{c}\,[\![\mathscr{P}^{\psi}]\!]\right) + \frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}\,[\![Q^{\psi}_{\|\varphi}]\!]\right) = -\,\varepsilon\, e\left[\!\!\left[\int F\,\frac{\partial\langle\Phi_{1\rm gc}\rangle}{\partial\varphi}\,d^3p\right]\!\!\right]. \tag{20}$$

Here, the total toroidal-momentum density $[\![P_{\|\varphi}]\!] + c^{-1}\,[\![\mathscr{P}^{\psi}]\!]$ is derived from the gyrocenter Vlasov moment of the toroidal gyrocenter velocity

$$\frac{\partial\mathbf{X}}{\partial\varphi}\cdot m\,\frac{d_{\rm gy}\mathbf{X}}{dt} \equiv mR^2\,\frac{d_{\rm gy}\varphi}{dt} = p_{\|}\,b_{\varphi} - \frac{\nabla\psi}{B\Omega}\cdot\left(\mu\,\nabla B + \frac{p_{\|}^2}{m}\,\widehat{\mathsf{b}}\cdot\nabla\widehat{\mathsf{b}} + \varepsilon\, e\nabla\langle\Phi_{1\rm gc}\rangle\right),$$

where the first term contributes to $[\![P_{\|\varphi}]\!]$ while the second set of terms contribute to the radial gyrokinetic polarization $[\![\mathscr{P}^{\psi}]\!] \equiv [\![\mathscr{P}^{\psi}_{\rm gc}]\!] + \varepsilon\,[\![\mathscr{P}^{\psi}_{\rm gy}]\!]$. As our last step, we perform a guiding-center multipole expansion: $\partial\langle\Phi_{1\rm gc}\rangle/\partial\varphi \equiv \partial\Phi_1/\partial\varphi + \langle\rho_{\rm gc}\rangle\cdot\nabla(\partial\Phi_1/\partial\varphi) + \cdots$ so that we find

$$e\int F\frac{\partial\langle\Phi_{1\rm gc}\rangle}{\partial\varphi}d^3p \simeq \left(\rho_{\rm gy} - \nabla\cdot\mathbb{P}_{\rm gc}\right)\frac{\partial\Phi_1}{\partial\varphi} + \nabla\cdot\left(\mathbb{P}_{\rm gc}\frac{\partial\Phi_1}{\partial\varphi}\right) = \varepsilon\left(\nabla\cdot\mathbb{P}_{\rm gy}\right)\frac{\partial\Phi_1}{\partial\varphi} + \nabla\cdot\left(\mathbb{P}_{\rm gc}\frac{\partial\Phi_1}{\partial\varphi}\right),$$

where we used the gyrokinetic quasineutrality condition $\rho_{\rm gy} - \nabla\cdot\mathbb{P}_{\rm gc} = \varepsilon\nabla\cdot\mathbb{P}_{\rm gy}$. Hence, the parallel-toroidal momentum equation (20) becomes

$$\frac{\partial}{\partial t}\left[\!\!\left[P_{\|\varphi} + \frac{1}{c}\,\mathscr{P}^{\psi}\right]\!\!\right] = -\,\frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}\left[\!\!\left[Q^{\psi}_{\|\varphi} + \varepsilon\,\mathscr{P}^{\psi}_{\rm gc}\frac{\partial\Phi_1}{\partial\varphi}\right]\!\!\right]\right) - \varepsilon^2\left[\!\!\left[\left(\nabla\cdot\mathbb{P}_{\rm gy}\right)\frac{\partial\Phi_1}{\partial\varphi}\right]\!\!\right]. \tag{21}$$

We note that, in the zero-Larmor-radius approximation $\mathbb{P}_{\rm gy} \simeq (mnc^2/B^2)\,\nabla_{\perp}\Phi_1$, we find

$$\left[\!\!\left[\left(\nabla\cdot\mathbb{P}_{\rm gy}\right)\frac{\partial\Phi_1}{\partial\varphi}\right]\!\!\right] \simeq \left[\!\!\left[\nabla\cdot\left(\mathbb{P}_{\rm gy}\frac{\partial\Phi_1}{\partial\varphi}\right) - \frac{\partial}{\partial\varphi}\left(\frac{mnc^2}{2\,B^2}|\nabla_{\perp}\Phi_1|^2\right)\right]\!\!\right] = \frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}\left[\!\!\left[\mathscr{P}^{\psi}_{\rm gy}\frac{\partial\Phi_1}{\partial\varphi}\right]\!\!\right]\right),$$

so that (21) becomes the gyrokinetic toroidal angular-momentum conservation law for the truncated gyrokinetic Vlasov-Poisson equations (compare with equation 98 of [2])

$$\frac{\partial}{\partial t}\left[\!\!\left[P_{\|\varphi} + \frac{1}{c}\,\mathscr{P}^{\psi}\right]\!\!\right] = -\,\frac{1}{\mathscr{V}}\frac{\partial}{\partial\psi}\left(\mathscr{V}\left[\!\!\left[Q^{\psi}_{\|\varphi} + \varepsilon\,\mathscr{P}^{\psi}\frac{\partial\Phi_1}{\partial\varphi}\right]\!\!\right]\right), \tag{22}$$

which includes guiding-center and gyrocenter polarization effects.

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References

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