Multiple Island Chains in Primary Resonances

M. C. de Sousa[1], I. L. Caldas[1], A. M. Ozorio de Almeida[2], F. B. Rizzato[3], R. Pakter[3]

[1] *Instituto de Física, Universidade de São Paulo, São Paulo, Brazil*

[2] *Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil*

[3] *Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil*

We analyse the dynamics of a relativistic particle moving in a uniform magnetic field and perturbed by a stationary electrostatic wave. We show that a pulsed wave produces an infinite number of perturbing terms with the same winding number. The perturbation coupling alters the number of island chains as a function of the parameters of the wave. We also observe that the number of chains in is always even if the number of islands in each chain is odd.

When we add a small perturbation to an integrable system, most of its periodic orbits with rational winding number are destroyed and just a few remain in phase space [1-3]. According to the Poincaré-Birkhoff Fixed Point Theorem, for each rational winding number, there is an even number of periodic trajectories, half stable and half unstable [1-3].

Resonant islands appear around the stable periodic points, and the islands surrounding the points of one orbit form an island chain [1-3]. The Poincaré-Birkhoff Fixed Point Theorem makes no claim on the number of chains for each rational winding number [1-3]. Although some twist systems present more than one chain [4, 5], the literature generally shows systems with just one island chain [1].

In this paper, we analyse the behaviour of a relativistic particle moving under the combined action of a uniform magnetic field and a stationary electrostatic wave. We show that a pulsed wave presents an infinite number of perturbing terms with the same rational winding number. For each winding number, the parameters of the wave determine which of these terms generate islands in the same region of phase space. As a consequence, the number of chains for the present system varies as a function of the wave number and wave period [6].

Consider a relativistic particle with charge q, rest mass m and canonical momentum $\mathbf{p}$ moving in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, with vector potential $\mathbf{A} = B_0 x\hat{\mathbf{y}}$, and perturbed by a stationary electrostatic wave with wave vector $\mathbf{k}$, period T and amplitude $\varepsilon / 2$ lying along the x axis. Using dimensionless quantities and making the canonical transformation

$x = \sqrt{2I}\sin\theta$ and $p_x = \sqrt{2I}\cos\theta$, the dimensionless Hamiltonian that describes the dynamics transverse to the magnetic field is given by [6]

$$H(I,\theta,t) = \sqrt{1+2I} + \frac{\varepsilon}{2}\cos(k\sqrt{2I}\sin\theta)\sum_{n=-\infty}^{+\infty}\delta(t-nT), \quad (1)$$

where the pulsed wave is described by the periodic sum of delta function kicks.

To determine the primary resonances, we expand (1) in a Fourier-Bessel series as

$$H = \sqrt{1+2I} + \frac{\varepsilon}{2T}\sum_{s=-\infty}^{+\infty}\cos\left(k\sqrt{2I}\sin\theta - \frac{2\pi st}{T}\right),$$

$$H = \sqrt{1+2I} + \frac{\varepsilon}{2T}\sum_{s=-\infty}^{+\infty}\sum_{r=-\infty}^{+\infty}J_r(k\sqrt{2I})\cos\left(r\theta - \frac{2\pi st}{T}\right), \quad (2)$$

where $J_r(k\sqrt{2I})$ are Bessel functions of order r.

From (2), we calculate the approximate resonant condition for the system

$$\frac{d}{dt}\left(r\theta - \frac{2\pi s t}{T}\right) = 0 \quad \Rightarrow \quad r\frac{d\theta}{dt} = \frac{2\pi s}{T} \quad \Rightarrow \quad r\omega_0 \cong s\omega, \quad (3)$$

where $\omega = 2\pi/T$ is the frequency of the electrostatic wave and we approximate $d\theta/dt \cong \omega_0$, with ω_0, the unperturbed frequency of the system, given by $\omega_0 = dH_0/dI = (1+2I)^{-1/2}$.

Since ω and ω_0 are both positive, (3) is satisfied only when r and s have the same sign. An analysis of the Hamiltonian (2) leads to the same conclusion, i.e., just the terms for which r and s have the same sign (with $r \neq 0$ and $s \neq 0$) are associated with resonances. Expression (3) gives the position of the resonances with respect to the action variable as

$$I_{r,s} \cong \frac{1}{2}\left(\frac{r}{s\omega}\right)^2 - \frac{1}{2} = \frac{1}{8}\left(\frac{rT}{s\pi}\right)^2 - \frac{1}{2}, \quad (4)$$

where $\Omega = s/r = \omega_0/\omega$ is the winding number, also known as rotation number, and for the resonances of the system Ω is rational.

Since the position $I_{r,s}$ is a function of the ratio $\Omega = s/r$ rather than of the individual numbers r and s, we notice that (2) presents an infinite number of perturbing terms that may generate islands in the same region of phase space. For example, the (1,1); (-1,-1); (2,2); (-2,-2), etc. perturbations are located in the same position $I_{1,1} \cong (T^2 - 4\pi^2)/8\pi^2$.

Considering r and s two positive relative primes, each (mr, ms) perturbation (with m integer and $m \neq 0$) may generate islands in the same region of phase space. For an isolated (mr, ms) resonance, there are $|m|$ chains characterized by $\Omega = s/r$. Each periodic orbit has r

points and between two consecutive pulses of the wave $\Delta\theta = 2\pi s/r = 2\pi\Omega \pmod{2\pi}$.

The superposition of an infinite number of perturbing terms that may generate islands in the same position alters the number of chains as a function of the parameters of the wave. Figure 1 shows the number of chains as function of the wave number k and wave period T for the (3,7) and (4,3) resonances. Each colour represents a different number of chains as indicated in the pictures. In both panels, we observe that the number of chains increases with k and T.

The islands of the (4,3) resonance are generated by the terms in (2) for which r is a multiple of 4 and $s = 3r/4$. However, the amplitude of the various perturbing terms is not the same. When $k \to 0$ or $T \to 3\pi/2$, the argument of the Bessel functions $J_r(k\sqrt{2I_{4,3}})$ tends to zero. Therefore, only the amplitude of the functions $J_{\pm 4}$ is considerable and the (4,3) resonance presents a single island chain in phase space, as can be seen in Fig. 1(b). Increasing the values of k or T, the amplitude of the functions $J_{\pm 8}$ cannot be neglected anymore and the resonance presents two island chains. If we continue increasing the values of k or T, more perturbing terms should be taken into account and the number of chains increases as well. Thus, we observe that although Hamiltonian (2) presents an infinite number of $(4m,3m)$ perturbations, the number M of island chains is finite for given values of the wave number and wave period. These chains are due to the $(4m,3m)$ perturbing terms for which $m \le M$, since the amplitudes of the terms with $m > M$ are much smaller and they do not generate islands in phase space [6].

We also note that the black curves in both panels of Fig. 1 respect the same power law. The black curves are the boundaries separating two regions with a different number of chains. For each resonance, the boundaries are related to the amplitude of the resonant terms generating the islands and they decay as $1/\sqrt{2I_{r,s}}$, where $I_{r,s}$ is given by expression (4).

Figure 1(a) shows that the (3,7) resonance always presents an even number of chains. It is possible to demonstrate that the total number of islands in phase space is always even [6]. Thus, the number of chains must be even for the resonances that present an odd number of islands in each of its chains. On the other hand, the number of chains may be even or odd when the number of islands in each chain is even, as can be seen in Fig. 1(b) for the (4,3) resonance.

We analysed the dynamics of a relativistic particle moving in a uniform magnetic field and perturbed by a stationary electrostatic wave. When the wave is given as a series of periodic pulses, we observe that the system presents an infinite number of perturbing terms with the same winding number. This superposition alters the number of island chains according to the parameters of the wave.

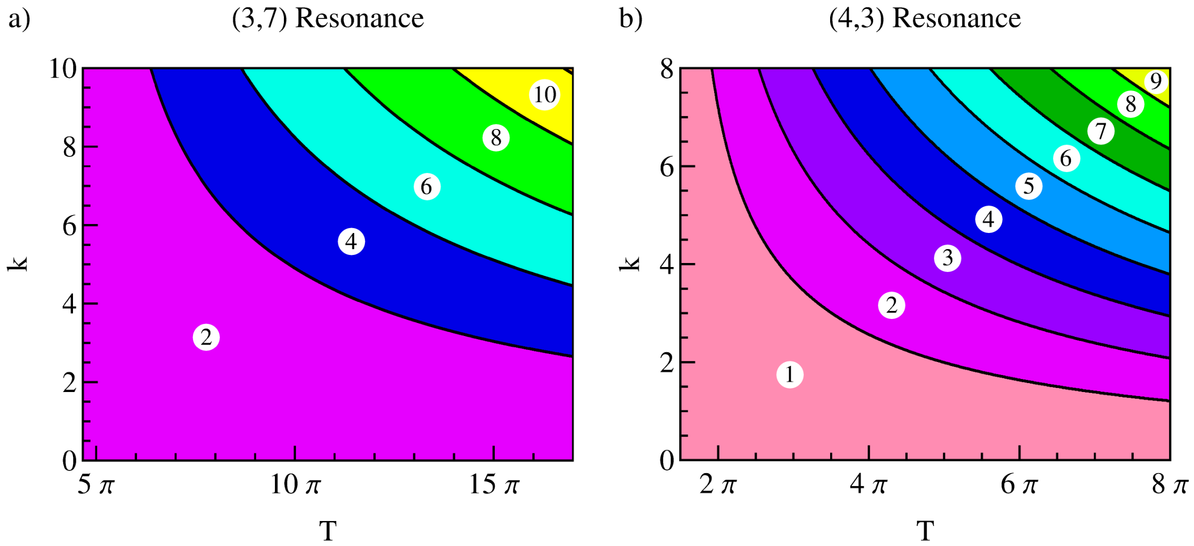

Figure 1 (Colour online): Number of chains as a function of k and T for the a) (3,7) and b) (4,3) resonances. Each colour represents a different number of chains as indicated in the pictures. Since the action is positive definite, the (3,7) and (4,3) resonances are not present in phase space for $T < 14\pi / 3$ and $T < 3\pi / 2$, respectively.

We built the parameter space showing the number of chains as a function of the wave number and wave period. For all the resonances, we observed that the boundaries separating two regions with a different number of chains respect the same power law. This power law is related to the amplitudes of the resonant terms that generate the islands of the resonance [6].

Another interesting feature of our system is that the number of chains is always even for the resonances that present an odd number of islands in each chain [6]. On the other hand, when the number of islands in each chain is even, the number of chains may be even or odd.

We point out that such results are not commonly observed in the literature. It holds for systems that are perturbed by an infinite number of terms with the same winding number. However, current studies suggest that even systems that have just a finite number of perturbing terms may present characteristics similar to the ones we observe in our system.

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