Basic microscopic plasma physics unified and simplified by N-body classical mechanics

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Abstract: The derivation of Debye shielding and Landau damping from the N-body description of plasmas requires many pages of heavy kinetic calculations in classical textbooks and is done in distinct, unrelated chapters. Using Newton's second law for the N-body system, we perform this derivation in a few steps with elementary calculations using standard tools of calculus, and no probabilistic setting. Unexpectedly, Landau damping turns out to be one facet of Debye shielding. This shielding and collisional transport are discovered to be two related aspects of the repulsive deflections of electrons. Using the shielded potential, the collisional diffusion coefficient is computed for the first time by a convergent expression including the correct calculation of deflections for all impact parameters. This paper provides a sketch of the corresponding derivations which are available in [1, 2].

Fundamental equation for the potential: This paper mainly deals with the One Component Plasma (OCP) model, which considers the plasma as infinite with spatial periodicity L in three orthogonal directions with coordinates (x,y,z), and made up of N electrons in each elementary cube with volume L^3. Ions are present only as a uniform neutralizing background, enabling periodic boundary conditions. The discrete Fourier transform of the electrostatic potential, readily obtained from the Poisson equation, is given by $\tilde{\varphi}(\mathbf{0}) = 0$, and for $\mathbf{m} \neq \mathbf{0}$ by

$$\tilde{\varphi}(\mathbf{m}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2} \sum_{j \in S} \exp(-\mathrm{i}\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j), \tag{1}$$

where $-e$ is the electron charge, ε_0 is the vacuum permittivity, $\mathbf{r}_j$ is the position of particle j, $S = \{1, \ldots N\}$, $\tilde{\varphi}(\mathbf{m}) = \int \varphi(\mathbf{r}) \exp(-\mathrm{i}\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}) \mathrm{d}^3\mathbf{r}$, with $\mathbf{m} = (m_x, m_y, m_z)$ a vector with three integer components running from $-\infty$ to $+\infty$, $\mathbf{k}_{\mathbf{m}} = \frac{2\pi}{L}\mathbf{m}$, and $k_{\mathbf{m}} = \|\mathbf{k}_{\mathbf{m}}\|$.

Let $\mathbf{r}_{l0}$ and $\mathbf{v}_l$ be the initial position and velocity of particle l, and let $\delta\mathbf{r}_l = \mathbf{r}_l - \mathbf{r}_{l0} - \mathbf{v}_l t$. In the following, we consider the $\delta\mathbf{r}_l$'s to be small. Therefore we approximate $\tilde{\varphi}_l(\mathbf{m})$ by $\tilde{\phi}_l(\mathbf{m})$, its expansion to first order in the $\delta\mathbf{r}_l$'s. We further consider φ to be small, and the $\delta\mathbf{r}_l$'s to be of the order of φ. We now introduce the time Laplace transform which transforms a function $f(t)$ into $\hat{f}(\omega) = \int_0^\infty f(t) \exp(\mathrm{i}\omega t) \mathrm{d}t$ (with ω complex). We Laplace transform both the potential and

Newton's equations of motion for the electrons. Combining the resulting equations yields

$$k_{\mathbf{m}}^2 \hat{\tilde{\phi}}(\mathbf{m},\omega) - \frac{e^2}{L^3 m_e \varepsilon_0} \sum_{\mathbf{n}} \mathbf{k_m} \cdot \mathbf{k_n} \sum_{j \in S} \frac{\hat{\tilde{\phi}}(\mathbf{n}, \omega + \omega_{\mathbf{n},j} - \omega_{\mathbf{m},j})}{(\omega - \omega_{\mathbf{m},j})^2} \exp[\mathrm{i}(\mathbf{k_n} - \mathbf{k_m}) \cdot \mathbf{r}_{j0}] = k_{\mathbf{m}}^2 \hat{\tilde{\phi}}^{(0)}(\mathbf{m},\omega), \tag{2}$$

where tildes (resp. carets) indicate the Fourier (resp. Laplace) transformed versions of the quantities, $\omega_{\mathbf{l},j} = \mathbf{k_l} \cdot \mathbf{v}_j$, and $\hat{\tilde{\phi}}_l^{(0)}(\mathbf{m},\omega)$ is the Laplace transform of $\tilde{\varphi}(\mathbf{m})$ computed by substituting $\mathbf{r}_l$ with its ballistic approximation $\mathbf{r}_{l0} + \mathbf{v}_l t$ in Eq. (1). Equation (2) is the fundamental equation of this paper. *This fundamental equation is of the type $\mathscr{E}\hat{\tilde{\phi}}$ = source term*, where $\mathscr{E}$ is a linear operator, acting on the infinite dimensional array whose components are all the $\hat{\tilde{\phi}}(\mathbf{m},\omega)$'s.

Shielded Coulomb potential: We introduce a smooth function $f(\mathbf{r},\mathbf{v})$, the *smoothed* velocity distribution function at $t = 0$. We assume it to be a spatially uniform distribution function $f_0(\mathbf{v})$ plus a small perturbation of the order of ϕ. We replace the discrete sums over particles in Eq. (2) by integrals over $f(\mathbf{r},\mathbf{v})$, and we keep the lowest order term in ϕ. Then operator $\mathscr{E}$ becomes diagonal with respect to both $\mathbf{m}$ and ω, and Eq. (2) becomes

$$\varepsilon(\mathbf{m},\omega)\hat{\tilde{\Phi}}(\mathbf{m},\omega) = \hat{\tilde{\phi}}^{(0)}(\mathbf{m},\omega), \tag{3}$$

where Φ is the new approximation of ϕ, and

$$\varepsilon(\mathbf{m},\omega) = 1 - \frac{e^2}{L^3 m_\mathrm{e} \varepsilon_0} \int \frac{f_0(\mathbf{v})}{(\omega - \mathbf{k_m} \cdot \mathbf{v})^2} \, \mathrm{d}^3\mathbf{v}. \tag{4}$$

This shows that the smoothed self-consistent potential $\hat{\tilde{\Phi}}$ is determined by the response function $\varepsilon(\mathbf{m},\omega)$. The latter is the classical plasma dielectric function.

By inverse Fourier-Laplace transform, to lowest order the contribution of particle j to $\Phi(\mathbf{r})$ turns out to be, after some transient whose duration is estimated later with the intuitive description of shielding, the *shielded Coulomb potential* of particle j

$$\delta\Phi_j(\mathbf{r}) = \delta\Phi(\mathbf{r} - \mathbf{r}_{j0} - \mathbf{v}_j t, \mathbf{v}_j), \tag{5}$$

where

$$\delta\Phi(\mathbf{r},\mathbf{v}) = -\frac{e}{L^3 \varepsilon_0} \sum_{\mathbf{m} \neq \mathbf{0}} \frac{\exp(\mathrm{i}\mathbf{k_m} \cdot \mathbf{r})}{k_{\mathbf{m}}^2 \, \varepsilon(\mathbf{m}, \mathbf{k_m} \cdot \mathbf{v})}. \tag{6}$$

Therefore, after this transient, *the dominant contribution to the full potential in the plasma turns out to be the sum of the shielded Coulomb potentials of individual particles located* at their ballistic positions.

Debye shielding and Landau damping: We now apply the smoothing using distribution function f to $\hat{\tilde{\phi}}^{(0)}(\mathbf{m},\omega)$ too in equation (3). To lowest order in the $\delta\mathbf{r}_j$'s, this approximates $\hat{\tilde{\Phi}}(\mathbf{m},\omega)$ by

$$\hat{\tilde{\Phi}}^{(0)}(\mathbf{m},\omega) = -\frac{\mathrm{i}e}{\varepsilon_0 k_{\mathbf{m}}^2}\int \frac{\tilde{f}(\mathbf{m},\mathbf{v})}{\omega - \mathbf{k}_{\mathbf{m}}\cdot\mathbf{v}}\,\mathrm{d}^3\mathbf{v}. \tag{7}$$

This shows that this second smoothing makes Eq. (3) *to become the expression including initial conditions in Landau contour calculations of Langmuir wave growth or damping*, usually obtained by linearizing Vlasov equation and using Fourier-Laplace transform, as described in many textbooks. Therefore, in these calculations, $\hat{\tilde{\Phi}}^{(0)}(\mathbf{m},\omega)$ turns out to be the smoothed version of the actual shielded potential in the plasma.

Intuitive interpretation of Debye shielding: Applying Picard iteration technique to the mechanical description of microscopic dynamics with the full OCP Coulomb potential of Eq. (1) yields the following interpretation of shielding for a particle in the bulk of the distribution function. At $t = 0$, consider a set of (uniformly, independently) randomly distributed particles. Consider a particle l. At a later time t, it has deflected all particles which made a closest approach to it with an impact parameter $b < v_{\mathrm{th}}t$ where v_{th} is the thermal velocity. This part of their global deflection due to particle l reduces the number of particles inside the sphere $S(t)$ of radius $v_{\mathrm{th}}t$ about it. Therefore the effective charge of particle l as seen out of $S(t)$ is reduced: the charge of particle l is shielded due to these deflections. This shielding effect increases with t, and thus with the distance to particle l. As a result, the typical time-scale for shielding to set in, when starting from random particle positions, is the time for a thermal particle to cross a Debye sphere, i.e. ω_{p}^{-1}, which sets the duration of the transients occurring in the inverse Laplace transform leading to Eq. (5); this order of magnitude is correct for a plasma close to equilibrium. Furthermore, *shielding is a cooperative dynamical process: it results from the accumulation of almost independent repulsive deflections with the same qualitative impact on the effective electric field of particle l* (if ions were added, the attractive deflection of charges with opposite signs would have the same effect). It is a cooperative effect, but not a collective one (it does not involve any synchronized motion of particles). Basic plasma physics textbooks show the accumulation of almost independent repulsive deflections to produce collisional transport of particles in plasmas. *Unexpectedly, it turns out that Debye shielding is another aspect of the same two-body repulsive process.*

Collisional transport: Collisional transport is described in textbooks with two opposite points of view. First, the two-body Rutherford collision picture which describes correctly collisions for impact parameters $b \ll d$, the interparticle distance; transport coefficients are computed by an ad hoc extension of the integrals over b up to the Debye length $\lambda_D \gg d$, and involve the Coulomb logarithm as a factor with some uncertainty. Second, a mean-field approach based on the Balescu-Lenard equation which describes correctly collisions for $b \gg d$; transport coefficients are computed by an ad hoc extension of the integrals over b down to λ_{ma}, the classical distance of minimum approach which is much smaller than d, and involve the Coulomb logarithm as a factor with some uncertainty. The agreement between the two calculations of transport coefficients gave confidence in their result, but till now no description of collisional transport has been describing correctly the scales about d.

We provide the first description of this kind by substituting the Coulomb potential in the One Component Plasma (OCP) model with the shielded potential of Eq. (5). Collisions with impact parameters $b \gg \lambda_{\text{ma}}$ can be described by lowest order perturbation theory. The corresponding deflection of a given particle turns out to be the sum of two-body Rutherford deflections with all other particles, as provided by lowest order perturbation theory. This provides a natural matching with the genuine two-body Rutherford deflection corresponding to $b \ll d$. Therefore *all impact parameters are correctly described, the integrals over b do not involve ad hoc cutoffs, and are convergent thanks to shielding*. One recovers the classical formulas for the transport coefficients with a Coulomb logarithm modified by a term of order 1 having a weak dependence on the particle velocity.

Conclusion: This new approach provides a unification and a simplification of basic microscopic plasma physics by a straightforward use of N-body classical mechanics. An old dream comes true: this mechanics can describe non trivial aspects of the macroscopic dynamics of a many-body system. The present theory has two natural extensions [1]. First, the derivation of the fundamental equation for the potential can be modified to enable the description of trapping or chaos due to Langmuir waves. This brings a natural link with previous works on wave-particle interaction [3, 4]. Second, there is a (more intricate) version of the fundamental equation (2) that does not involve linearization. It might be used to study the effect of the coupling of Fourier components with both coherent and incoherent effects.

References

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