Global geodesic acoustic mode in a tokamak

with positive magnetic shear profile

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One of the novel and most prominent results in the theory of geodesic acoustic modes (GAMs) discovered in numerical simulations is the existence of a global GAM (GGAM) [1]. Such mode can appear as a discrete eigenmode in the gap of the continuous spectrum and has finite spacial width. The importance of global mode study is associated with the fact that these particular modes (in contrary to the modes of the continuous spectrum) are thought to "survive" and to be observed under the real conditions in dissipative medium. Moreover, the independence of the frequency of GAM on plasma radius observed in some recent experiments – see, e. g., Refs. [2, 3] – serves as a direct evidence of the global structure of geodesic acoustic mode. In Ref. [1] GGAMs have been found only for negative shear discharges when local GAM frequency, $\omega_{\text{GAM}}^2 = \omega_s^2(2 + 1/q^2)$, has a maximum inside the plasma (here ω_s is the frequency of sound, q is the safety factor).

In this paper we derive analytically the condition of the appearance of global GAM and present the indicative example of the GGAM for a tokamak with positive magnetic shear profile. We use the standard reduced one-fluid MHD model in the electrostatic approximation:

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{c}\mathbf{j} \times \mathbf{B}_0,$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \mathbf{v} = 0,$$

$$\mathbf{v} = \frac{c}{B_0^2}\mathbf{B}_0 \times \nabla \phi + v_{||}\frac{\mathbf{B}_0}{B_0},$$

$$j_{||} = -\frac{c}{4\pi}\nabla_\perp^2 A_{||}, \quad \nabla \cdot \mathbf{j} = 0,$$

where the usual notations are used; subscript "0" denotes the equilibrium quantities. Looking for axisymmetric solutions and excluding $A_{||}$, $\mathbf{j}$ and $\mathbf{v}$, we arrive to the following set of coupled equations for the poloidal Fourier harmonics of the potential and plasma pressure:

$$\omega^2 \left\{ \frac{1}{r}\frac{d}{dr}\left(\frac{r}{c_A^2}\frac{d\phi_m}{dr}\right) - \frac{m^2}{r^2 c_A^2}\phi_m \right\} - \frac{m^2}{qR^2}\left\{ \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\frac{\phi_m}{q}\right) - \frac{m^2}{r^2}\frac{\phi_m}{q} \right\} - \tag{1}$$

$$- \frac{4\pi\omega}{cRB_s}\left(\frac{dp_{m+1}}{dr} - \frac{dp_{m-1}}{dr} + \frac{m+1}{r}p_{m+1} + \frac{m-1}{r}p_{m-1} \right) = 0,$$

$$\left(\omega^2 - \frac{m^2 c_s^2}{q^2 R^2}\right) p_m + \frac{cm\omega}{rB_s}\frac{dp_0}{dr}\phi_m + \tag{2}$$

$$+ \frac{c\rho_0 c_s^2 \omega}{RB_s}\left(\frac{d\phi_{m+1}}{dr} - \frac{d\phi_{m-1}}{dr} + \frac{m+1}{r}\phi_{m+1} + \frac{m-1}{r}\phi_{m-1}\right) = 0.$$

Here $c_A = B_a/\sqrt{4\pi\rho_0}$, where $B_a = B_{0|r=0}$; $c_s = (\gamma p_0/\rho_0)^{1/2}$; m is the poloidal wavenumber. Low-pressure plasma equilibrium in a large aspect ratio tokamak with circular magnetic surfaces is assumed (r – radius of the magnetic surface counted from the magnetic axis). Keeping in Eqs. (1) – (2) the zeroth harmonic of potential, ϕ_0, and the first harmonics of pressure, $p_{\pm 1}$, we easily obtain the ordinary GAM spectrum. To obtain a global solution we add into consideration the second harmonics of electrostatic potential, $\phi_{\pm 2}$. Note that the first harmonics of the pressure enter in Eq. (1) solely in the combination ($p_1 - p_{-1}$) determined by the ϕ_0: $p_1 - p_{-1} = Rc\rho_0\omega(d\phi_0/dr)/B_a$. Introducing normalized radius $\hat{r} = r/a$ and frequency $\hat{\omega}^2 = \omega^2 R^2/c_{s|\hat{r}=0}^2$, we arrive to the system of two equations for ϕ_0 and $\Phi_2 = \hat{r}^2(\phi_2 + \phi_{-2})$:

$$\left(\hat{\omega}^2 - T\left(2 + \frac{1}{q^2}\right)\right)\frac{d\phi_0}{d\hat{r}} + \frac{T}{\hat{r}^2}\frac{d\Phi_2}{d\hat{r}} = 0, \tag{3}$$

$$\frac{\hat{\omega}^2 \beta}{4}\frac{d}{d\hat{r}}\left(\frac{1}{\hat{r}}\frac{d\phi_0}{d\hat{r}}\right) - \frac{d}{d\hat{r}}\left(\frac{1}{\hat{r}^3}\left[\frac{1}{q^2} - \frac{\hat{\omega}^2 \beta}{4}\right]\frac{d\Phi_2}{d\hat{r}}\right) - \frac{\Phi_2}{\hat{r}^3 q^3}\left\{\frac{3}{\hat{r}}\frac{dq}{d\hat{r}} - \frac{d^2 q}{d\hat{r}^2} + \frac{2}{q}\left(\frac{dq}{d\hat{r}}\right)^2\right\} = 0. \tag{4}$$

Here $T(\hat{r})$ is a normalized temperature equal to the unity at $\hat{r} = 0$, and $\beta = c_s^2/c_{A|\hat{r}=0}^2$; ρ_0 is assumed to be constant for a simplicity. The second term in the square brackets in Eq. (4) is negligible with respect to the first one and, therefore, may be omitted. Below we also omit hats on $\hat{r}$ and $\hat{\omega}$ operating with normalized radius and normalized frequency only.

Eq. (4) can be integrated in an elementary way under condition $3qdq/dr - rqd^2q/dr^2 + 2r(dq/dr)^2 = 0$, which uniquely determines the radial profile of safety factor: $q = q_0 q_1/(q_1 - (q_1 - q_0)r^4)$. Here $q_0 = q_{|r=0}$ and $q_1 = q_{|r=1}$. In what follows we work with such a choice of q-profile only. If $q_1 > q_0$, this profile describes monotonic growth of q with small gradient near the axis. After integration of Eq. (4), we obtain the following expressions for $d\phi_0/dr$ and $d\Phi_2/dr$:

$$\frac{d\phi_0}{dr} = \frac{KrTq^2}{\omega^2(1 + \beta Tq^2/4) - T(2 + 1/q^2)}, \tag{5}$$

$$\frac{d\Phi_2}{dr} = -\frac{Kr^3 q^2(\omega^2 - T(2 + 1/q^2))}{\omega^2(1 + \beta Tq^2/4) - T(2 + 1/q^2)}, \tag{6}$$

where K is the constant of integration defining the amplitude of the considered modes.

The existence of a global mode is determined by two conditions. First, the continuously differentiable solution can be found if there is no singularities in the right-hand sides of Eqs. (5)

– (6). In our case, this condition reduces to the positivity of denominator, $\omega^2(1 + \beta T q^2/4) - T(2 + 1/q^2) > 0$, that should be satisfied at every point in plasma. Second, the global mode solution has to obey the boundary conditions. We impose zeroth boundary conditions for Φ_2: $\Phi_{2|r=0} = 0$, $\Phi_{2|r=1} = 0$, which can be summarized in the single integral requirement for the existence of global GAM:

$$\int_0^1 d\Phi_2/dr\, dr = 0 \quad \text{or} \quad \int_{q_0}^{q_1} \frac{(\omega^2 - T(2 + 1/q^2))dq}{\omega^2(1 + \beta T q^2/4) - T(2 + 1/q^2)} = 0. \tag{7}$$

It can be easily understood that Eq. (7) determines the eigenfrequency of the GGAM.

To perform the integration in Eq (7), we consider the representative class of temperature profiles in the form $T = q^2/(A_0 + A_2 q^2 + A_4 q^4)$. For these profiles, condition (7) reduces to the simple combination of two integrals:

$$q_1 - q_0 + \frac{\beta}{4A_4}\left(c \int_{q_0}^{q_1} \frac{dq}{q^4 + bq^2 + c} + b \int_{q_0}^{q_1} \frac{q^2 dq}{q^4 + bq^2 + c}\right) = 0, \tag{8}$$

where $b = (A_2 - 2/\omega^2)/(A_4 + \beta/4)$, $c = (A_0 - 1/\omega^2)/(A_4 + \beta/4)$. The expressions of the incoming integrals in elementary functions are determined by the sign of the parameter $\delta = b^2 - 4c$, therefore, there are two forms of the dispersion relation for GGAMs:

for $\delta < 0$,

$$q_1 - q_0 - \frac{\beta}{8A_4\sqrt{-\delta}}\left\{ \frac{(b - \sqrt{c})c_+}{2} \ln\left|\frac{q^2 + qc_- + \sqrt{c}}{q^2 - qc_- + \sqrt{c}}\right|\Bigg|_{q_0}^{q_1} - \right. \tag{9}$$

$$\left. - c_-\left[b\left(\text{arctg}\left(\frac{2q + c_-}{c_+}\right)\Bigg|_{q_0}^{q_1} + \text{arctg}\left(\frac{2q - c_-}{c_+}\right)\Bigg|_{q_0}^{q_1}\right) + \sqrt{c}\,\text{arctg}\left(\frac{q^2 - \sqrt{c}}{qc_+}\right)\Bigg|_{q_0}^{q_1}\right]\right\} = 0;$$

for $\delta > 0$,

$$q_1 - q_0 - \frac{\beta}{8A_4\sqrt{\delta}}\left\{(bb_- - c)I(b_-) - (bb_+ - c)I(b_+)\right\} = 0, \tag{10}$$

$$\text{where } I(b_\pm) = \begin{cases} \frac{2}{\sqrt{b_\pm}}\text{arctg}\left(\frac{q}{\sqrt{b_\pm}}\right)\Big|_{q_0}^{q_1}, & b_\pm > 0 \\[2ex] \frac{1}{\sqrt{-b_\pm}}\ln\left|\frac{\sqrt{-b_\pm} - q}{\sqrt{-b_\pm} + q}\right|\Big|_{q_0}^{q_1}, & b_\pm < 0 \end{cases}$$

For brevity, we use designations: $b_\pm = (b \pm \sqrt{\delta})/2$, $c_\pm = \sqrt{2\sqrt{c} \pm b}$.

Eq. (9) with $\delta < 0$ corresponds to the regimes with a maximum of the local GAM frequency, ω_{GAM}, within the plasma column. In the considered case of monotonically growing q with radius, this maximum can be provided by the temperature profile with a non-axis maximum

(e. g., as a result of a powerful non-axis auxillary plasma heating). The radial structure of the mode is similar to the structure of the solution derived numerically in Ref. [1] for the discharges with the reverse magnetic shear.

Eq. (10) with $\delta > 0$ takes place for monotonically decreasing temperature. Note that b_- is always negative, therefore, there is always a logarithmic branch in $I(b_\pm)$. The eigenfrequency satisfying Eq. (10) provides the argument of the logarithm be close to zero that needs very precise fit of the frequency. That is why this solution is difficult to be found by direct numerical calculations. Formally, such solution always exists, but in practise it takes place only for temperature profiles with rather flat region in the plasma core and if the range of q variation in the plasma is relatively small. On Fig. 1a, we present the example of the solution obtained in the case $\delta > 0$ for T- and q-profiles shown on Fig. 1b; $\beta = 0.04$. Thus, we demonstrate analytically the possibility for GGAM formations in the discharges with monotonic profiles of the local GAM frequency.

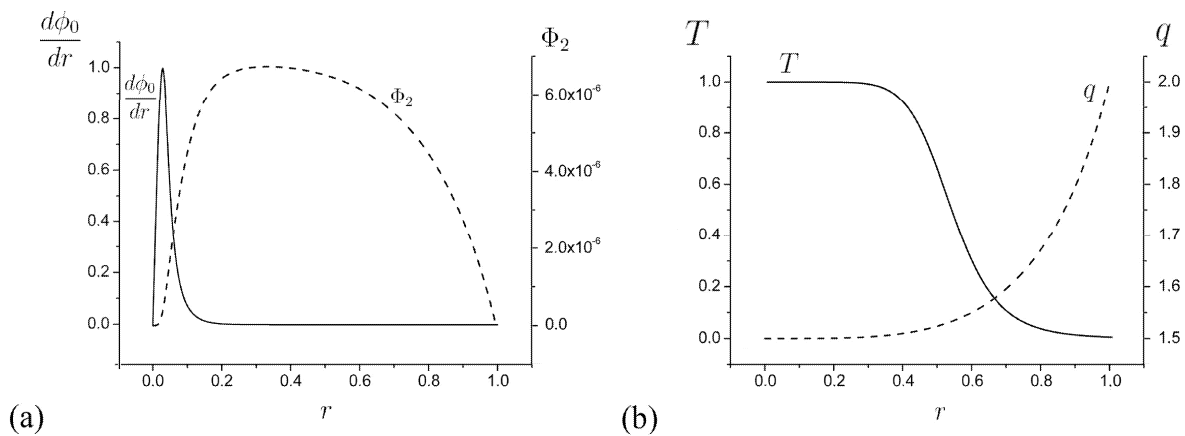(a)   (b)

Figure 1: (a) – radial profiles of $d\phi_0/dr$ (solid line) and Φ_2 (dashed line); (b) – radial profiles of the temperature (solid line) and of the safety factor (dashed line). Here $\omega^2 = 2.39$.

References

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