Shearing modes approach in the theory of shear flows turbulence

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It is well known that the presence of a strong radial electric field in tokamaks and stellarators influences plasma turbulence, transport, and related phenomena. The spontaneous appearance of the strong radially inhomogeneous electric field in the edge layer of the tokamak plasma leads to the poloidal rotation with radially inhomogeneous flow velocity. The low-frequency plasma turbulence, and therefore the anomalous heat conduction and particle diffusivity, appear suppressed by shearing flow that results in the formation of improved energy-confinement regimes in tokamak plasmas (e.g. the H-mode regime). The theory of plasma instabilities and turbulence are grounded on the application of the spectral transforms over the time and spatial coordinates and on the investigation of the spectral properties, the stability, and the temporal evolution of separate spectral harmonics. However, it is worth noting that the analytical description of the stability and turbulence of the magnetized plasma in strong inhomogeneous electric field can suffer from inadequate application of the spectral transforms to shearing plasma flow having an inhomogeneous radial profile of velocity. In typical treatment of kinetic theory of magnetized plasma in the presence of this inhomogeneous electric field, a transformation to the reference frame that moves with the non-uniform equilibrium fluid velocity is performed without unchanging the spatial (laboratory) coordinates. As a result, the spatial inhomogeneity, inherent in the flow-velocity profile, remains in the Vlasov equation. Different approximations, some dubious, have been used in the application of the Fourier transforms to Vlasov and Maxwell equations. Here, we illustrate that the application such "physical" approximations, sometimes used for the performance of the spectral transforms of such equations, invalidate the results obtained. We argue that the transformation, to convective reference frame, of both velocity and spatial coordinates in the Vlasov equation is necessary for the proper treatment of the inhomogeneity introduced by external electric field in the Vlasov equation,

$$\frac{\partial F_\alpha}{\partial t}+\hat{\mathbf{v}}\frac{\partial F_\alpha}{\partial \hat{\mathbf{r}}}+\frac{e}{m_\alpha}\left(\mathbf{E}_0\left(\hat{\mathbf{r}}\right)+\frac{1}{c}\left[\hat{\mathbf{v}}\times\mathbf{B}\right]-\bigtriangledown\varphi\left(\hat{\mathbf{r}},t\right)\right)\frac{\partial F_\alpha}{\partial \hat{\mathbf{v}}}=0. \tag{1}$$

and for performing the proper spectral transforms without invalid approximations.

We use a slab geometry with the mapping $(r,\theta,\varphi)\rightarrow(\hat{x},\hat{y},\hat{z})$ where r,θ,φ are the radial, poloidal and toroidal directions, respectively, of the toroidal coordinate system. We consider the case of shearing plasma flow in linearly changing electric field $\mathbf{E}_0(\hat{\mathbf{r}}) = (\partial E_0/\partial \hat{x})\,\hat{x}\mathbf{e}_x$ with $\partial E_0/\partial \hat{x} = const$. In that case

$$\mathbf{V}_0(\mathbf{r}) = V_0(\hat{x})\,\mathbf{e}_y = -\frac{c}{B_0}\frac{\partial E_0}{\partial \hat{x}}\hat{x}\mathbf{e}_y = V_0'\hat{x}\mathbf{e}_y \tag{2}$$

with spatially homogeneous, $V_0' = const$, velocity shear. The spatially homogeneous part of flow's velocity shear is eliminated from the problem by a simple Galilean transformation. It was obtained in Ref.[1], that the transition in the Vlasov equation from velocity $\hat{\mathbf{v}}$ and coordinates $\hat{x}, \hat{y}, \hat{z}$ to convected coordinates $\mathbf{v}$ in velocity space, determined by $\hat{v}_x = v_x,\ \hat{v}_y = v_y + V_0'x,\ \hat{v}_z = v_z$ and to sheared flow coordinates x, y, z in the configurational space, determined by $\hat{x} = x,\ \hat{y} = y + V_0'tx,\ \hat{z} = z$ (it is assumed that the inhomogeneous electric field, and consequently the shear in the flow, originate at time $t = t_{(0)} = 0$) transforms the linearised Vlasov equation for $f_\alpha = F_\alpha - F_{0\alpha}$, with known equilibrium distribution $F_{0\alpha}$, to the form,

$$\frac{\partial f_\alpha}{\partial t} + v_{\alpha x}\frac{\partial f_\alpha}{\partial x} + \left(v_{\alpha y} - v_{\alpha x}V_0't\right)\frac{\partial f_\alpha}{\partial y} + v_{\alpha z}\frac{\partial f_\alpha}{\partial z} + \omega_{c\alpha}v_{\alpha y}\frac{\partial f_\alpha}{\partial v_{\alpha x}}$$
$$-\left(\omega_{c\alpha} + V_0'\right)v_{\alpha x}\frac{\partial f_\alpha}{\partial v_{\alpha y}} = \frac{e_\alpha}{m_\alpha}\left(\frac{\partial \varphi}{\partial x} - V_0't\frac{\partial \varphi}{\partial y}\right)\frac{\partial F_{0\alpha}}{\partial v_{\alpha x}} + \frac{e_\alpha}{m_\alpha}\frac{\partial \varphi}{\partial y}\frac{\partial F_{0\alpha}}{\partial v_{\alpha y}} + \frac{e_\alpha}{m_\alpha}\frac{\partial \varphi}{\partial z_\alpha}\frac{\partial F_{0\alpha}}{\partial v_{\alpha z}}, \tag{3}$$

(ω_c is the cyclorton frequency of ion (electron)), which is free from the spatial inhomogeneities originated from shear flow (see also Eq.(8) in Ref.[1]), where $F_{0\alpha}$ is assumed as inhomogeneous over coordinate x. In that case, the Fourier transformation of the Vlasov equation over spatial sheared coordinates is performed exactly without application of the dubious approximations such as the "slow" spatial variation of the flow velocity. With velocity coordinates $v_\perp$, ϕ

$$v_x = v_\perp\cos\phi, \quad v_y = \sqrt{\eta}v_\perp\sin\phi, \quad \phi = \phi_1 - \sqrt{\eta}\omega_c t, \quad v_z = v_z, \tag{4}$$

where $\eta = 1 - V_0'/\omega_c$, and with modified gyro-frequency $\sqrt{\eta}\omega_c$, and with leading gyro-center coordinates X, Y, determined in convective-shearing coordinates by the relations $x = X - \frac{v_\perp}{\sqrt{\eta}\omega_c}\sin\phi$, $y = Y + \frac{v_\perp}{\eta\omega_c}\cos\phi + V_0't\,(X - x)$, $z_1 = z - v_z t$, the Vlasov equation (1), in which species index is suppressed, transforms to the form [1]

$$\frac{\partial F}{\partial t} + \frac{e}{m\eta\omega_c}\left(\frac{\partial \varphi}{\partial X}\frac{\partial F}{\partial Y} - \frac{\partial \varphi}{\partial Y}\frac{\partial F}{\partial X}\right) + \frac{e}{m}\frac{\sqrt{\eta}\omega_c}{v_\perp}\left(\frac{\partial \varphi}{\partial \phi_1}\frac{\partial F}{\partial v_\perp} - \frac{\partial \varphi}{\partial v_\perp}\frac{\partial F}{\partial \phi_1}\right) - \frac{e}{m}\frac{\partial \varphi}{\partial z_1}\frac{\partial F}{\partial v_z} = 0, \tag{5}$$

in which all time-dependent coefficients are absent. With leading center coordinates the Fourier transform for the electrostatic potential becomes

$$\varphi(x,y,z,t) = \int \varphi(k_x,k_y,k_z,t)\,e^{ik_xx+ik_yy+ik_zz}dk_xdk_ydk_z = \int \varphi(k_x,k_y,k_z,t)$$
$$\times\exp\left[ik_xX_i + ik_yY_i + ik_zz - i\frac{\hat{k}_\perp(t)\,v_\perp}{\sqrt{\eta}\omega_{ci}}\sin\left(\phi_1 - \sqrt{\eta}\omega_{ci}t - \theta(t)\right)\right]dk_xdk_ydk_z, \tag{6}$$

where $\hat{k}_{\perp}^{2}(t) = \left(k_x - V_0' t k_y\right)^2 + \frac{1}{\eta} k_y^2$, and $\tan\theta = k_y / \sqrt{\eta}(k_x - V_0' t k_y)$. It follows from Eq.(6), that the only manifestation of the shearing flow in Vlasov equation is in the time dependence of the finite Larmor radius effect. Therefore this effect is the basic linear mechanism of the action of the velocity shear on waves and instabilities in shearing plasma flow[1, 2]. It was obtained in [1, 2], that this effect leads to the non-modal decrease of the frequency and growth rate of the unstable electrostatic drift perturbations with time.

Usually, however, only the transformation to the coordinate frame in velocity space, that moves with flow velocity, but unchanged in configuration space, is usually used in kinetic theory of plasma shear flows. After that transformation to convective frame the linearized Vlasov equation becomes

$$\frac{\partial f_\alpha}{\partial t} + V_0' \hat{x} \frac{\partial f_\alpha}{\partial \hat{y}} + \hat{\mathbf{v}} \frac{\partial f_\alpha}{\partial \hat{\mathbf{r}}} + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - \left(\omega_{c\alpha} + V_\alpha'\right) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} = \frac{e_\alpha}{m_\alpha} \bigtriangledown \varphi\left(\hat{\mathbf{r}}, t\right) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}. \quad (7)$$

Traditionally, in proceeding with derivation of the governing equation for f_α, the spatial Fourier transform in laboratory configuration space,

$$\varphi\left(\hat{x}, \hat{y}, \hat{z}, t\right) = \int \varphi\left(\hat{k}_x, \hat{k}_y, \hat{k}_z, t\right) e^{i\hat{k}_x \hat{x} + i\hat{k}_y \hat{y} + i\hat{k}_z \hat{z}} d\hat{k}_x d\hat{k}_y d\hat{k}_z, \quad (8)$$

for the electrostatic potential φ and for f_α is adapted with assumption of the "slow variation" of $V_0(x)$ with spatial coordinates. That gives the following equation for $f_\alpha\left(t, \hat{\mathbf{k}}, \mathbf{v}\right)$:

$$\frac{\partial f_\alpha}{\partial t} - i\hat{k}_y V_0(\hat{x}) f_\alpha - i\left(\hat{\mathbf{k}}\hat{\mathbf{v}}\right) f_\alpha + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - \left(\omega_{c\alpha} + V_\alpha'\right) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}} = -\frac{e_\alpha}{m_\alpha} . \hat{\mathbf{k}} \varphi\left(\hat{\mathbf{k}}, t\right) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}.$$

Then, velocity shear is absorbed into the identical for both plasma species Doppler shifted frequency $\hat{\omega} = \omega - k_y V_0(\hat{x})$, and, in fact, is excluded from the subsequent analysis.

If we apply, however, the Fourier transform directly to (7) over the spatial coordinates in the laboratory frame without the application of the approximation of the "slow" spatial variation of the flow velocity, we receive the following equation for $f_\alpha\left(t, \hat{\mathbf{k}}, \mathbf{v}\right)$:

$$\frac{\partial f_\alpha}{\partial t} - V_0' \hat{k}_y \frac{\partial f_\alpha}{\partial \hat{k}_x} - i\left(\hat{\mathbf{k}}\hat{\mathbf{v}}\right) f_\alpha + \omega_{c\alpha} v_{\alpha y} \frac{\partial f_\alpha}{\partial v_{\alpha x}} - \left(\omega_{c\alpha} + V_0'\right) v_{\alpha x} \frac{\partial f_\alpha}{\partial v_{\alpha y}}$$
$$= i \frac{e_\alpha}{m_\alpha} \hat{\mathbf{k}} \varphi\left(\hat{\mathbf{k}}, t\right) \frac{\partial F_{0\alpha}}{\partial \hat{\mathbf{v}}}. \quad (9)$$

For the deriving from Eq.(9) the equation which couples f_α and φ of the separate spatial Fourier mode, as it is in Eq.(5), we have to exclude from Eq.(9) the term $-V_0' \hat{k}_y \frac{\partial f_\alpha}{\partial \hat{k}_x}$, due to which the Fourier mode of f_α appears to be coupled with all Fourier modes of the electrostatic potential and depends on the integral of φ over wave-number space. The characteristic equation $dt = -\frac{d\hat{k}_x}{V_0' \hat{k}_y}$ gives the solution $\hat{k}_x + V_0' t \hat{k}_y = K_x$, where K_x as the integral of this equation is time independent. It reveals that $f_\alpha = f_\alpha\left(K_x, \hat{k}_y, \hat{k}_z, t\right) = f_\alpha\left(\hat{k}_x + V_0' t\hat{k}_y, \hat{k}_y, \hat{k}_z, t\right)$, i.e. the wave number components $\hat{k}_x$ and $\hat{k}_y$ have to be changed in such a way that $\hat{k}_x + V_0' t\hat{k}_y$ leaves unchanged with time. If we use $\hat{k}_x = K_x - V_0' t\hat{k}_y$ in Eqs.(8) and (9), we obtain for the electrostatic potential the presentation (6), and we obtain Eq.(5) for f_α, with time independent $K_x = k_x$, $\hat{k}_y = k_y$, $\hat{k}_z = k_z$. The obtained results prove, that the solution of the Vlasov equation for times $t > \left(V_0'\right)^{-1}$ in the form of the separate Fourier harmonic with time independent wave numbers may be obtained only in convected-sheared coordinates. That solution reveals in the laboratory frame as a shearing mode

$$\varphi\left(\hat{x}, \hat{y}, \hat{z}, t\right) = \int \varphi\left(k_x, k_y, k_z, t\right) e^{i\left(k_x - V_0' t k_y\right)\hat{x} + ik_y\hat{y} + ik_z\hat{z}} dk_x dk_y dk_z. \tag{10}$$

with time dependent x-component of the wave number.

The oversimplification of the problem, which resulted from the application of the assumption of "slow" spatial variation of flow velocity, leads to the overlooking of that principal effect of shear flow. It is obvious, that the time dependence in (3) may be neglected only in the case of negligible velocity shear, or when the very short evolutionary time is considered. Really, for $k_x \sim k_y$ and with velocity shearing rate V_0' of the order of the growth rate γ of the instability considered, for time $t > \gamma^{-1}$, we have $V_0' t k_y > k_x$ in the integral $K_x = \hat{k}_x + V_0' t\hat{k}_y$. Therefore the assumption of "slow spatial variation of flow velocity" is not valid for the investigations of the effects of shear flow in real experiments, where observed velocity shearing rate may be of the order or above of the instability growth rate and the time of the observations is of the order of the inverse growth rate or longer.

The joint transformation of the velocity and spatial coordinates in Vlasov equation from laboratory to convected-sheared coordinates in the case of strong shear flow in the crossed magnetic and inhomogeneous electric field, is the unavoidable for the proper treating the spectral properties of plasma in inhomogeneous electric field.

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