Blended high-order finite element methods for plasma modeling

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Abstract

Blended finite element (FE) methods combine high-order continuous and discontinuous spatial representations to exploit the expected physical behavior of the plasma dynamics. High-order accurate FE methods benefit problems that have strong anisotropies, complicated geometries, or stiff governing equations. The method is applicable to continuum models, such as continuum kinetic and multi-fluid plasma models. Discontinuous and blended FE methods are implemented in a flexible code framework, WARPX. The algorithm is applicable to study advanced physics calculations of plasma dynamics including HEDP, magnetic plasma confinement, and astrophysical plasmas.

Owing to the complexity of plasma phenomena, a thorough understanding requires validated physical models, verified computational simulations, and well-diagnosed experiments. This paper focuses on developing computational methods for plasma models with sufficient physical and numerical fidelity to generate insight and predictability.

Continuum Plasma Models

Discrete models that account for each constituent particle is not particularly useful for the numerical treatment of realistic plasmas where the number of particles (N) and the number of interactions ($> N^2$) is not computationally tractable. Instead an ensemble average is performed to give a statistical description. Plasmas may be most accurately modeled using kinetic theory, where distribution functions, $f_s(\mathbf{x},\mathbf{v})$, are governed by a Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = \left.\frac{\partial f_s}{\partial t}\right|_{collisions} \tag{1}$$

for each species s. Combined with Maxwell's equations, the system leads to the *continuum kinetic plasma model*. Kinetic models in their most general form are six-dimensional, but reduced models, e.g. gyrokinetic, can also be meaningful. Further reduced plasma models result by taking moments over velocity space of Eq. (1) and of f_s, the *multi-fluid plasma model*.[1]

The principal variables of the multi-fluid plasma model are derived from moments of the distribution functions. The $5M$ model directly evolves the variables given by the first two moments and the tensor contraction of the third moment,

$$p_s = \rho_s T_s = \frac{1}{3} m_s \int w^2 f_s(\mathbf{v}) d\mathbf{v}, \tag{2}$$

where T_s is the temperature. The $13M$ model[2] directly evolves the variables given by the first three moments and the tensor contraction of the fourth moment,

$$\mathbf{h}_s = \frac{1}{2} m_s \int w^2 \mathbf{w} f_s(\mathbf{v}) d\mathbf{v}. \tag{3}$$

The system of moment equations is truncated, retaining only variables with a physical meaning, and closed by relating higher moment variables to lower moment variables.

Fluids are coupled to each other and to the fields through Maxwell's equations and interaction source terms. Including elastic scattering and reacting collisions (e.g. ionization, recombination, charge exchange) introduces additional source terms. For example, the electron momentum equation for an interacting three-fluid model (electron, ion, neutral) is given by

$$\begin{aligned} \frac{\partial \rho_e \mathbf{u}_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e \mathbf{u}_e + p_e \mathbf{I} + \Pi_e) = & -\frac{\rho_e e}{m_e} (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \mathbf{R}_i^{ie} + \mathbf{R}_e^{en} \\ & + m_e \mathbf{u}_n \Gamma_i^{ion} - m_e \mathbf{u}_e \Gamma_n^{rec} \end{aligned} \tag{4}$$

where the source and sink rates are computed by the appropriate convolution integrals, such as the ion source rate from electron impact ionization with neutrals

$$\Gamma_i^{ion} = \int f_n(\mathbf{v}') \int f_e(\mathbf{v}) \sigma_{ion} \left(|\mathbf{v} - \mathbf{v}'| \right) |\mathbf{v} - \mathbf{v}'| \, d\mathbf{v} d\mathbf{v}'. \tag{5}$$

See Ref. [3] for the details of the $5M$ multi-fluid plasma model. Additional moment models can be found in Refs. [1, 4, 5, 6].

The governing equations for the $5M$ or $13M$ model can be expressed in balance law form as

$$\frac{\partial}{\partial t} q + \nabla \cdot F = S, \tag{6}$$

where q is the vector of conserved variables, F is the flux tensor, and S is vector of source terms. Eigenvalues of the flux and source Jacobians ($\partial F / \partial Q$, $\partial S / \partial Q$) provide the characteristic speeds and frequencies. Table 1 shows typical time scales for a laboratory FRC plasma and an ionospheric F region plasma. The large time scale separation makes the system mathematically stiff and complicates accurate solution.

Table 1: Typical plasma time scales (secs)

	FRC	F Region
$1/\omega_{pe}$	5×10^{-14}	6×10^{-8}
L/c	3×10^{-9}	7×10^{-2}
$1/\omega_{ci}$	10^{-8}	4×10^{-3}
L/v_A	10^{-5}	3×10^{1}
τ_{eq}	10^{-3}	10^{5}

Finite Element Methods

Finite element (FE) methods offer high-order spatial accuracy in an unsplit approach that tightly couples the flux and source terms in Eq. (6). FE methods are appropriate for problems that have strong anisotropies, complicated geometries, or stiff governing equations. Magnetized plasma simulations of realistic devices using the continuum kinetic or the multi-fluid plasma models benefit from high-order accuracy. FE methods expand the solution vector using a set of basis functions, $v_k(\mathbf{r})$, and project the governing equations onto the same basis functions using a Galerkin method. Integrating by parts and applying the divergence theorem gives

$$\int_\Omega v_k \frac{\partial q}{\partial t} dV + \oint_{\partial\Omega} v_k \mathbf{F} \cdot d\mathbf{S} - \int_\Omega \mathbf{F} \cdot \nabla v_k dV = \int_\Omega v_k S dV \tag{7}$$

The domain is divided into finite elements, and the integral equation is applied to each element with some assumption of continuity at the element boundaries.

If the solution is assumed to be continuous, the result is the usual finite element method, which works well for many elliptic and parabolic systems on complicated geometries; however, spurious oscillations can occur at discontinuities (shocks) for hyperbolic systems, which means the method is not suitable for many plasma simulations.

If the solution is allowed to be discontinuous, but with continuous fluxes as required by the conservation law, the resulting finite element system is the discontinuous Galerkin (DG) method.[7, 8] Fluxes in the surface integral term in Eq. (7) are evaluated using an upwind method such as an approximate Riemann solver[4, 9].

$$\mathbf{F}_{\partial\Omega} = \frac{1}{2}\left(\mathbf{F}^+ + \mathbf{F}^-\right) - \frac{1}{2}\sum_k l_k \left(q^+ - q^-\right) |\lambda_k| \, r_k \tag{8}$$

Limiting is accomplished by locally reducing the expansion order. Time is advanced using a Runge-Kutta method, for example the third-order, TVD method.[10]

Blended FE methods combine high-order continuous and discontinuous spatial representations to exploit the expected physical behavior of the plasma dynamics with improved computational efficiency. The blended FE method has been implemented in a flexible code framework, WARPX (Washington Approximate Riemann Plasma code)[1]. The code runs on multiprocessor machines using MPI and on GPU systems using OpenCL.

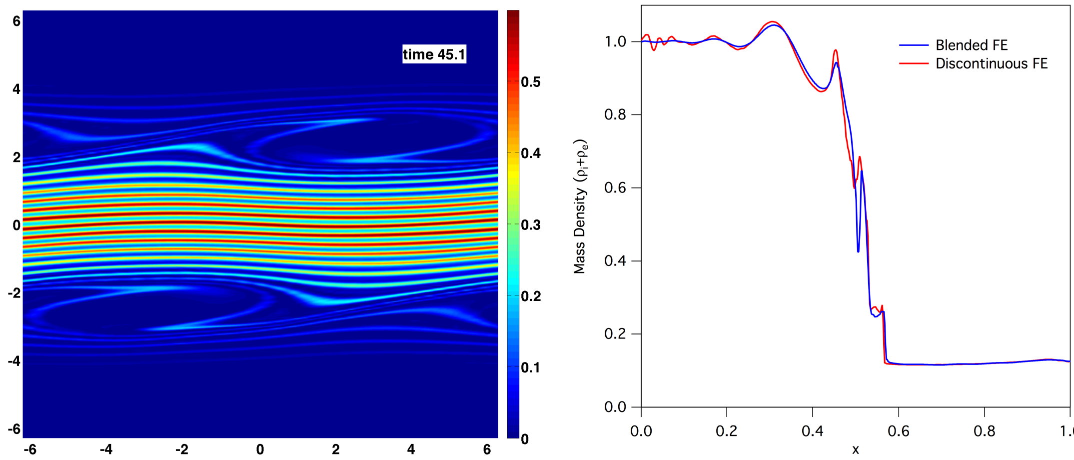
Numerical Results

The DG method has been applied to solve the Vlasov-Poisson model. The solution of the continuum kinetic model has been benchmarked to weak and strong Landau damping and to the two-stream instability.[11] Figure 1(a) shows the simulation results at $\omega_p t = 60$, using 20×80 elements and 7^{th} order polynomials in (x, v_x). The high-order representation accurately captures the fine-scale striations the occur in phase space.

The blended FE method has been applied to the two-fluid, electromagnetic plasma shock problem[4], a generalization of the MHD shock problem[12]. A continuous FE representation is applied to the electron fluid and to the electric and magnetic fields, and a discontinuous FE representation is applied to the ion fluid. The solution using 2^{nd} order elements with 512 elements is shown in Fig. 1(b). The solution compares well with the solution from a DG method.

Conclusions

Plasmas are accurately described with continuum models: from detailed kinetic models to $5M$ and $13M$ moment models. This paper highlights developments in high-order techniques for



(a) (b)

Figure 1: Numerical results: (a) Strong Landau damping using 7th order discontinuous FE representation to accurately capture fine-scale striations in phase space. (b) Electromagnetic shock tube problem solved with the blended finite element method.

solving these systems of equations so as to create high-fidelity methods that are better able to capture complex plasma physics phenomena. Coupling these models to high-order spatial representations leads to numerical algorithms that capture appropriate physical phenomena. Specifically, the continuous and discontinuous FE methods have been blended to provide a physics-based numerical approach. The blended FE method and the various plasma models have been implemented into the WARPX code and have been validated to analytical results and benchmarked to published computational results.

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