Solution of the Grad-Shafranov equation for the helical states of the RFX-mod experiment

G. L. Trevisan, P. Zanca and D. Terranova

Consorzio RFX, Associazione EURATOM-ENEA sulla Fusione, Padova, Italy

Abstract

An iterative procedure for the solution of the helical Grad-Shafranov equation is introduced, and the application of the algorithm to the quasi-helical states of the RFX-mod experiment is presented and discussed.

Introduction

Grad-Shafranov's equation is a two-dimensional differential equation useful to describe MHD equilibria and often exploited in conjunction with external magnetic measurements in order to model internal profiles. It can be derived for a symmetric configuration, that could be either an axisymmetric toroidal system or, as in this paper, a helical-symmetric cylindrical system. Since helical states are naturally and routinely found in RFP dynamics as a meaningful example of its dynamo-like self-organizing nature [1], the exploitation of the helical Grad-Shafranov's equation in the modelling of the equilibrium is a natural way of linking a fully three-dimensional configuration with a two-dimensional equation [2].

Helical states of the RFX-mod experiment

Generally speaking, the RFP configuration is characterized by the concurrent presence and interplay of a wide range of MHD modes, in both the poloidal mode number m and the toroidal mode number n and thus the 'helicity' m/n. Such a configuration, dubbed 'Multiple Helicity' (MH), has poor confinement properties and exhibits enhanced transport due to magnetic chaos.

In recent years [1] a new kind of paradigm for the RFP has emerged, following both theoretical analyses and experimental studies: it has been dubbed 'Quasi Single Helicity' (QSH) configuration, and is characterized by a MHD perturbation spectrum where a single helicity mode, together with its higher order harmonics, can be found to be much higher in amplitude than the other modes. In a QSH state magnetic chaos is found to be reduced leading therefore to better confinement properties and reduced transport.

Finally, one can easily think of a fully 'Single Helicity' (SH) configuration where there is a dominant mode (with its higher order harmonics) but there are no secondary modes at all. In a SH state the configuration is symmetric with respect to the helicity of the dominant mode. This is the configuration that will be discussed throughout the paper.

Helical Grad-Shafranov's equation

Following [2], let us start from a cylindrical coordinate system (r, ϑ, z), where the linear coordinate z describing the direction along the cylinder (that is, along the linearized torus) has period $2\pi R_0$ (R_0 is the major radius) and can thus be replaced by an angle-like coordinate ϕ.

The helical symmetry assumption is that in the MHD spectrum there is a single dominant m, n mode together with its higher order harmonics. It is straightforward to notice that the physical angular dependence is upon a 'helical angle' $u \equiv m\vartheta - n\phi$, not upon the two angle coordinates separately, and thus each quantity is completely determined by its (r, u) dependence and can therefore be decomposed as a Fourier series in terms of a single 'helical mode number' q:

$$\mathbf{B}(r,u) = \mathbf{B}^0(r) + \sum_q \mathbf{b}^q(r)\,\mathrm{e}^{iqu}. \tag{1}$$

In RFX-mod the most internally resonant mode, the $m = 1, n = 7$ tearing, is the dominant mode $q = 1$ that determines the helicity of the SH state, while its higher order harmonics correspond to higher q numbers (e.g.: $q = 2$ means $m = 2, n = 14$, etc). Harmonics higher than $q = 2$ have been proved to be negligible in the expansion in eq. (1) and have thus been neglected.

Since pressure effects are small in present RFPs, they have been neglected. The resulting force-free equation has to be solved together with Ampère's Law:

$$\mathbf{j} \times \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{B} = \mathbf{j}. \tag{2}$$

In terms of the 'helical flux' χ and the 'helical field' g,

$$\chi = mA_\phi + n\varepsilon A_\vartheta, \qquad g = mB_\phi + n\varepsilon B_\vartheta, \tag{3}$$

the magnetic field has the representation:

$$\mathbf{B}(r,u) = f(r)\nabla\chi(r,u) \times \boldsymbol{\sigma}(r) + f(r)g(r,u)\boldsymbol{\sigma}(r), \tag{4}$$

where $\varepsilon(r) = r/R_0$, $f(r) = r/(m^2 + n^2\varepsilon^2)$, $\boldsymbol{\sigma}(r) = \hat{r} \times \nabla u$. The Grad-Shafranov's equation derives from (2) and can be shown to take the helical form:

$$\frac{1}{f}\frac{\partial}{\partial r}\left(f\frac{\partial \chi}{\partial r}\right) + \frac{1}{rf}\frac{\partial^2 \chi}{\partial u^2} = \beta g(\chi) - g\frac{\mathrm{d}g}{\mathrm{d}\chi}, \qquad \text{(HGS)} \tag{5}$$

with $\beta = 2mn/R_0(m^2 + n^2\varepsilon^2)$. The helical field is proven to be a flux function, $g = g(\chi)$.

The eigenfunction $\chi(r,u)$ can be decomposed similarly to eq. (1), so that each Fourier harmonic χ^q can be solved through a separate differential equation:

$$\frac{1}{f}\frac{\mathrm{d}}{\mathrm{d}r}\left(f\frac{\mathrm{d}\chi^q}{\mathrm{d}r}\right) + \frac{(iq)^2}{rf}\chi^q = \beta g^q - \left(g\frac{\mathrm{d}g}{\mathrm{d}\chi}\right)^q, \qquad q = 0,1,2. \tag{6}$$

Once the helical flux is known, the magnetic field can be promptly computed through eq. (4).

Algorithm

In order to solve the HGS equation an iterative approach has been adopted: by separating the LHS from the RHS, one can solve eq. (6) as one ordinary differential equation for each harmonic χ^q, and eventually reach convergence and self-consistency. A Fortran code has been developed with this purpose.

As initialization, the $q=0$ field is solved using a force-free α-Θ_0 model [3]. Since pressure is neglected, the F and Θ constraints are not *exactly* matched. Then, the $q=1$ eigenfunction is computed solving a cylindrical Newcomb equation [4], that can be interpreted as the linear approximation of eq. (5) in a perturbative approach. For the dominant harmonic, $m=1$, $n=7$, two magnetic measurements are available in RFX-mod: the radial and toroidal components of the perturbed field on the inner side of the conductive shell, $b_r^{1,7}$ and $b_\phi^{1,7}$. They are both used to impose the boundary conditions (BCs) in order to get a meaningful eigenfunction. The higher order harmonic $q=2$ is neglected in the 'zeroth' iteration.

Then, the actual iterative procedure starts: first, flux surface averages $\langle\cdot\rangle_\chi$ are computed [5] in order to get a helical field as a flux function, $g(\chi)=\langle g(r,u)\rangle_\chi=\langle mB_\phi+n\varepsilon B_\vartheta\rangle_\chi$; next, the helical field is expressed again in terms of (r,u), as $g(r,u)=g(\chi(r,u))$; after that, the RHS of eq. (5) is computed by Fourier transform of the g and $g\,\mathrm{d}g/\mathrm{d}\chi$ terms; finally, the actual integration is carried out and new eigenfunctions χ^q are computed. For $q=0$ there is one BC, which has to be exploited to impose regularity constraints on the origin, while for $q=1,2$ there are two: one is used for regularity and the other to match external measurements ($b_r^{q=1}=b_r^{1,7}$, $b_r^{q=2}=0$). The procedure stops as soon as a few user-determined convergence criteria are met.

Convergence

A satisfying self-consistent solution has proven to be rather difficult to find. First of all, a back-averaging method had to be implemented in the code in order to successfully smooth the sharply varying features of the successive iterations, following the formula:

$$\chi_{(i+1)}=(1-K)\,\chi_{(i)}+K\,\chi_{(i-1)}, \tag{7}$$

where $\chi_{(i)}$ is the ith iteration solution and $K\approx 80\%$. This method slows down the iterative dynamics, since just $1-K\approx 20\%$ of new information is considered in each step, but at the same time favours convergence in the overall scheme [6].

A second aspect of convergence to be tackled with particular care is the problem of the BCs. When solving for the $q=1$ harmonic of the HGS eq. there is just one BC, $b_r=b_r^{1,7}$, to be imposed and therefore the solution $\chi_{\text{ode}}^{q=1}$ does not *exactly* satisfy the second one, $b_\phi=b_\phi^{1,7}$. By introducing an auxiliary solution, $\chi_{\text{aux}}^{q=1}$, that satisfies the 'ideal shell' ($b_r=0$) condition, one can promptly show that by combining both solutions:

$$\chi^{q=1} = \chi^{q=1}_{\text{ode}} + C_{\text{aux}}\, \chi^{q=1}_{\text{aux}}, \tag{8}$$

the resulting $\chi^{q=1}$ automatically satisfies the BC for b_r, and that the additional BC for b_ϕ can be enforced by appropriately choosing the C_{aux} coefficient.

Results

A simulation typically reaches convergence within a few tens of iterations. Fig. (1) shows that the auxiliary coefficient is found to be $C_{\text{aux}} \approx 4\%$, while the $\delta(i, i-1)$ variable, measuring the quadratic difference between two successive iterations, amounts to $\delta \approx 0.2\%$ upon convergence.

Fig. (1) also shows the final computed solution for $b_r^{q=1}$ together with the starting guess, that is, Newcomb's solution (the discontinuity in its first derivative is due to a singular term at the resonant radius in Newcomb's eq. [4] that is eventually smoothed away by the HGS eq.).

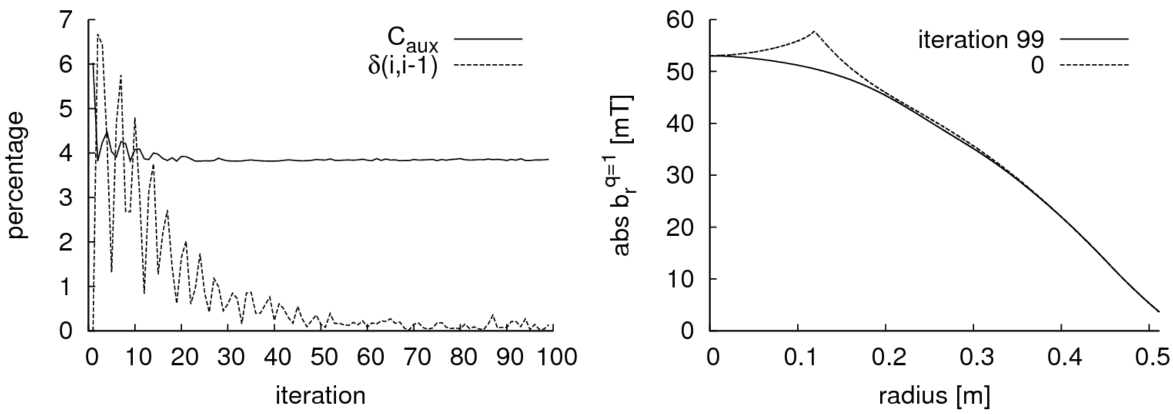

Figure 1: Typical iterating sequence and computed eigenfunction for an $m = 1, n = 7$ simulation.

The computed $\chi^{q=1}$ solution satisfies both BCs and is currently being benchmarked against the Variational Moments Equilibrium Code (VMEC) [7] with encouraging results.

References

[1] R. Lorenzini *et al.*, Nature Physics **5**, 570–574 (2009)

[2] J. M. Finn, R. Nebel and C. Bathke, Phys. Fluids B **4**, 1262 (1992)

[3] V. Antoni *et al.*, Nucl. Fusion **26**, 1711 (1986)

[4] R. Fitzpatrick, Phys. Plasmas **6**, 1168 (1999)

[5] W. D. D'Haeseleer *et al.*, New York: Springer-Verlag (1991)

[6] M. E. Kress, Journal of Computational Physics **76**, 201–230 (1988)

[7] S. P. Hirshman and J. C. Whitson, Phys. Fluids **26**, 3554 (1983)