Validity of the Landau-Lifshitz approximation in an ultra-high intensity laser pulse

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Introduction

The back-reaction on an accelerating electron of the radiation it emits—first analysed by Lorentz [1], Abraham [2] and Dirac [3]—gives rise to an equation of motion, now known as the Lorentz-Abraham-Dirac (LAD) equation, which reads

$$\ddot{x}^a = -\frac{q}{m}F^a{}_b\dot{x}^b + \tau\Delta^a{}_b\dddot{x}^b. \tag{1}$$

Here, q and m are the charge and mass and $\tau := q^2/6\pi m \simeq 6\times 10^{-24}$ s the characteristic time of the electron. An overdot denotes the proper time derivative, Δ is the $\dot{x}$-orthogonal projection, and F is the electromagnetic field.

The LAD equation is not regarded as a satisfactory equation, as its solutions possess unphysical features, such as exponentially growing accelerations ("runaway solutions"). Such solutions can be eliminated, but only at the cost of violating causality.

A common approach to overcoming the difficulties of the LAD equation is to treat the radiation reaction as a small perturbation to the applied Lorentz force, and keep only terms up to order τ [4]. This yields a second order equation,

$$\ddot{x}^a = -\frac{q}{m}F^a{}_b\dot{x}^b - \tau\frac{q}{m}\dot{x}^c\partial_c F^a{}_b\dot{x}^b + \tau\frac{q^2}{m^2}\Delta^a{}_bF^b{}_cF^c{}_d\dot{x}^d, \tag{2}$$

known as the Landau-Lifshitz (LL) equation, which is free from the difficulties of runaway solutions and preacceleration that beset (1). However, in addition to requiring that the radiation reaction be small compared to the applied force, one might ask why we should trust such an approximation, if we don't trust the equation it is approximating.

An alternative description [5, 6] of radiation reaction was derived by Ford and O'Connell by assuming the electron has a non-pointlike structure. In the presence of an external force f^a, the Ford-O'Connell (FO) equation reads

$$\ddot{x}^a = f^a + \tau\Delta^a{}_b\dot{f}^b. \tag{3}$$

It is often claimed that the FO equation is identical to the LL approximation [7]. However, this is not the case if the external force depends on the velocity, as the Lorentz force does. Because the FO equation is regarded as exact, we cannot replace acceleration terms arising from the derivative of the applied force with the applied force itself, as is done in the LL prescription. However, since the FO equation also arises as an intermediate step in the derivation of LL, it can be used to test the validity of the latter.

In this paper, we explore the detailed form of the FO equation in an external electromagnetic field. We then compare its predictions with those of the LL equation for the case of a particle interacting with a harmonic plane wave, and find good agreement even where one might *a priori* expect the LL description to break down.

Ford-O'Connell equation

Using the Lorentz force $f^a = -\frac{q}{m}F^a{}_b\dot{x}^b$ as the applied force in (3), the Ford-O'Connell equation takes the form

$$M^a{}_b\ddot{x}^b = -\frac{q}{m}(F^a{}_b + \tau\dot{x}^c\partial_c F^a{}_b)\dot{x}^b, \tag{4}$$

where $M^a{}_b = \Delta^a{}_b + \tau G^a{}_b$ with $G^a{}_b = \frac{q}{m}\Delta_c F^c{}_d\Delta^d{}_b$. It can be shown that M has determinant

$$\det M = 1 + \frac{\tau^2}{2}G^{ab}G_{ab}. \tag{5}$$

Since $\frac{1}{2}G^{ab}G_{ab} = (\frac{q}{m}B)^2$ where B is the magnetic field "seen" by the particle, it follows that M is invertible, so the FO equation is a viable equation of the motion.

It is clear that, ignoring terms of order τ^2 or higher, the FO and LL equations agree. For the LL equation to be a good approximation to FO, then, it is necessary that

$$\tau\sqrt{\frac{1}{2}G^{ab}G_{ab}} \ll 1. \tag{6}$$

Particle motion in a harmonic plane wave

Radiation reaction effects will be important for high energy electrons interacting with ultra-intense laser pulses. For simplicity, we model the laser pulse by a harmonic plane wave:

$$\frac{q}{m}F^1{}_3 = \frac{q}{m}F^1{}_0 = \mathscr{E}\sin(k\phi), \tag{7}$$

where $\mathscr{E}$ is a constant and $\phi = x^0 + x^3$. All other $F^a{}_b = 0$.

Applying the harmonic plane wave (7) to the FO equation, and setting $\dot{x}^2 = 0$, yields

$$\ddot{\phi} = -\tau\mathscr{E}\sin(k\phi)\dot{\phi}^2\Big[\frac{(\mathscr{E}\sin(k\phi)\dot{\phi} + \tau k\mathscr{E}\cos(k\phi)\dot{\phi}^2)}{1 + \tau^2\mathscr{E}^2\sin^2(k\phi)\dot{\phi}^2}\Big], \tag{8}$$

$$\ddot{x}^1 = -\Big[\frac{(1 + \tau\mathscr{E}\sin(k\phi)\dot{\phi}\dot{x}^1)(\mathscr{E}\sin(k\phi)\dot{\phi} + \tau k\mathscr{E}\cos(k\phi)\dot{\phi}^2)}{1 + \tau^2\mathscr{E}^2\sin^2(k\phi)\dot{\phi}^2}\Big], \tag{9}$$

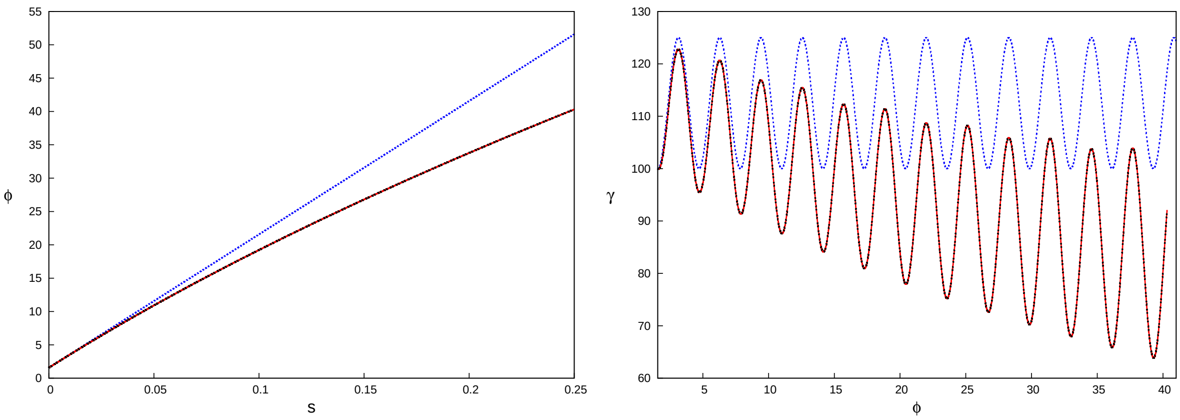

Figure 1: Radiation reaction effects of a pulse with $a_0 = 100$ on an electron of initial energy $\gamma_{\rm in} = 100$. Left panel: ϕ as a function of proper time s. Right panel: γ as a function of ϕ. Dotted blue curves without radiation reaction; solid red curves with LL radiation reaction; double-dotted black curves with FO radiation reaction.

and the electron's energy, normalized to mc^2, is given by

$$\gamma = \frac{1 + (\dot{x}^1)^2 + \dot{\phi}^2}{2\dot{\phi}}, \tag{10}$$

From (6), it follows that the LL equation should be reliable only when $\tau\sqrt{G^{ab}G_{ab}/2} \ll 1$, or in the plane wave (7),

$$\mathscr{T} := \tau\dot{\phi}\mathscr{E}\sin(k\phi) \ll 1. \tag{11}$$

As Fig. 1 shows, for the highest currently attainable laser intensities ($a_0 = 100$) and moderately high initial electron energies ($\gamma = 100$), radiation reaction has a significant effect, but LL and FO are in good agreement.

If we consider the most intense lasers under development ($a_0 \sim 1000$) and the highest energy electrons available ($\gamma \sim 10^5$), we appear to be in a regime where the condition (11) is violated, and we would expect the predictions of LL and FO to diverge. However, as shown in Fig. 2, although the dynamics is dominated by radiation reaction, agreement between the two theories remains good. How are we to explain this?

The condition (11) refers to the instantaneous energy and field strength, whereas the previously quoted values of a_0 and γ refer to the *peak field* and *initial energy*. From Fig. 2, it is clear that the electron almost instantaneously loses most of its energy to radiation. After this, its radiation is greatly reduced, and (11) is satisfied throughout the remainder of its evolution, ensuring that LL is reliable.

Conclusion

Radiation reaction undoubtedly has important effects on the motion of an electron subject to intense fields, but its theoretical description remains obscure. By analysing the interaction of

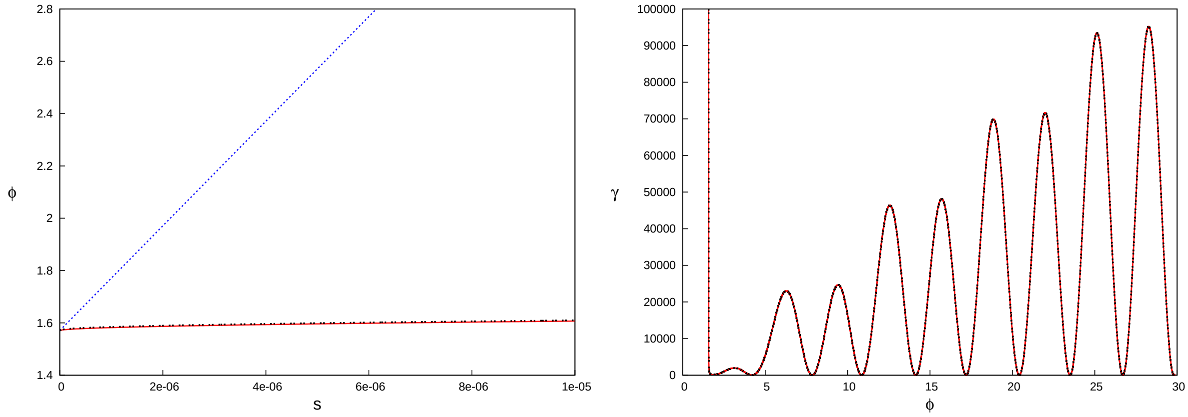

Figure 2: Radiation reaction effects of a pulse with $a_0 = 1000$ on an electron of initial energy $\gamma_{\text{in}} = 10^5$. Left panel: ϕ as a function of proper time s. Right panel: γ as a function of ϕ. Dotted blue curve without radiation reaction; solid red curves with LL radiation reaction; double-dotted black curves with FO radiation reaction.

a high energy electron with an intense laser pulse, we have shown that the radiation reaction effects ensure that it does not enter a regime where the LL description breaks down, even when *a priori* estimates would suggest otherwise.

It is worth noting that the current analysis is limited to classical dynamics in a harmonic plane wave, which may restrict its validity. Effects due to the pulse structure and quantum dynamics will be addressed in future work.

Acknowledgements

We are grateful to David Burton and Jonathan Gratus for many valuable discussions, and to Enrico Brunetti for technical assistance. We acknowledge support from SUPA, the Cockcroft Institute, the U.K. EPSRC, the EC's LASERLAB-EUROPE/LAPTECH (grant agreement no. 284464) and the Extreme Light Infrastructure (ELI) European Project.

References

[1] H. A. Lorentz, *The theory of electrons and its applications to the phenomena of light and radiant heat* (Stechert, New York, 1916).

[2] M. Abraham, *The classical theory of electricity and magnetism* (Blackie, London, 1932).

[3] P. A. M. Dirac, Proc. R. Soc. A **167** 148 (1938).

[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1962).

[5] G. W. Ford and R. F. O'Connell, Phys. Lett. A **157**, 217 (1991).

[6] G. W. Ford and R. F. O'Connell, Phys. Lett. A **174**, 182 (1993).

[7] R. F. O'Connell, Phys. Lett. A **313**, 491 (2003).