

Helically symmetric magnetic islands in tokamaks and negative shear configurations

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We start from a current-carrying cylindrical plasma with imposed periodicity $L_z = 2\pi R$ modeling a large aspect ratio toroidal configuration of major radius R . Applying a helical deformation of the equilibrium plasma boundary with the same helical pitch as magnetic field lines at $q = m/n$ results in the breaking of topology of the helical flux surfaces and the appearance of magnetic islands at the corresponding resonant surface. Such a helical equilibrium with islands is typically unstable against internal modes resonant with the island chain (helical wave number $n_h = 0$) [1]. While rotational symmetry restrictions on the plasma boundary shape exist for the islands at the rational magnetic surfaces with $q > 1$, any shaping of the plasma column, including spatial helical axis, is admissible to produce single helicity magnetic islands for $q = 1$ and $q = 1/n$ in general.

The stability of the chains of helically symmetric magnetic islands is investigated for tokamak-like configurations with positive shear and for the configurations with safety factor q decreasing to the plasma boundary (negative magnetic shear) relevant to RFP configurations [2]. The unstructured grid MHD_NX code is used to compute ideal MHD stability of 2D (cylindrical, axial or helical symmetry) equilibria with arbitrary topology of magnetic surfaces [3].

1 Helically symmetric equilibria with islands

The equilibrium magnetic field in helically symmetric plasma $\vec{B} = (\nabla\psi_h \times e_3 + fe_3)/g_{33}$ can be represented using the curvilinear coordinates $(x^1, x^2, x^3) = (u, v, z)$ with the corresponding contravariant $e_k = \partial\vec{r}/\partial x^k$ and covariant $e^k = \nabla x^k$ vectors. The $(u, v) = (r \cos \theta_h, r \sin \theta_h)$ plane is rotating about the origin according to the polar coordinate transformation (r, θ) into $(r, \theta_h = \theta - \kappa z)$, where the helix pitch is $2\pi/\kappa$.

The generalized Grad-Shafranov equilibrium equation for the helical flux ψ_h takes the form:

$$-\sum_{i,k=1}^2 \frac{\partial}{\partial x^i} \left(\frac{G_{ik}}{g_{33}} \frac{\partial \psi_h}{\partial x^k} \right) = p' + \frac{ff'}{g_{33}} - \frac{2\kappa f}{g_{33}^2}, \quad (1)$$

$$G_{11} = 1 + \kappa^2 v^2, \quad G_{12} = -\kappa^2 uv, \quad G_{22} = 1 + \kappa^2 u^2, \quad g_{33} = 1 + \kappa^2(u^2 + v^2).$$

We assume a linear dependence in ψ_h for the current density $j_3 = ff' + g_{33}p'$ and for force-free configurations ($p' = 0$), we have

$$j_3 = \alpha \psi_h / a^2 + A, \quad (2)$$

where a is the plasma minor radius and the coefficients α and A are varied to obtain a family of

equilibria with islands.

A standard model for a tokamak in the limit of large aspect ratio is 1D circular cylinder equilibrium with the safety factor $q = rB_z/RB_\theta$, where $0 < z < 2\pi R$, R is the major radius of the equivalent torus with aspect ratio $R/a \gg 1$. The values $R = 1$ and $a = 0.1$ are used; $f = 1$ is a good approximation for strong longitudinal magnetic field. Direct solutions of the equation (1) with perturbed boundary were investigated in [1]. However, due to the linearity of the equilibrium equation with the chosen r.h.s (2) and with the approximation $f = \text{const}$ a helically symmetric equilibrium with islands can be constructed as a sum of the cylindrically symmetric part satisfying the inhomogeneous equation (1) and a helical eigensolution (α, ψ_h) of the homogeneous equation without the additional helical term $-2\kappa f/g_{33}^2$ and with $A = 0$. Provided that the cylindrical solution features local extremum for the value of α equal to the eigenvalue (resonant α), the sum of the solutions gives the equilibrium with islands. In contrast to the case with non-resonant α when the islands can be generated only due to the boundary deformation, the island width in the resonant equilibria can be varied keeping the same circular boundary and varying the amplitude H of the helical eigenfunction normalized by the maximum of the cylindrically symmetric part. The local extrema in the helical flux function ψ_h correspond to the presence of the magnetic surface $q = 1/(R\kappa)$ inside the 1D cylindrical equilibrium. The value of A controls current density at the boundary and global shear. The equilibrium equation is linear in ψ_h and can be readily solved numerically. Increasing the values of A in (2) beyond the $2\kappa f$ lead to the change of sign of ψ_h and corresponds to hollow current density profile. In Fig.1 the flux functions ff' related to the longitudinal current density are plotted across the plasma midplane for different values of α and the coefficient A . For the helical pitch $\kappa = 1$ the value of $A = 2$ defines the limit between positive and negative shear (hollow current density profile) configurations for a chosen α due to the presence of the additional helical term. In the next section the ideal MHD stability of the equilibria with the $m = 2, 3, 4$ island chains and with different current density profiles is presented.

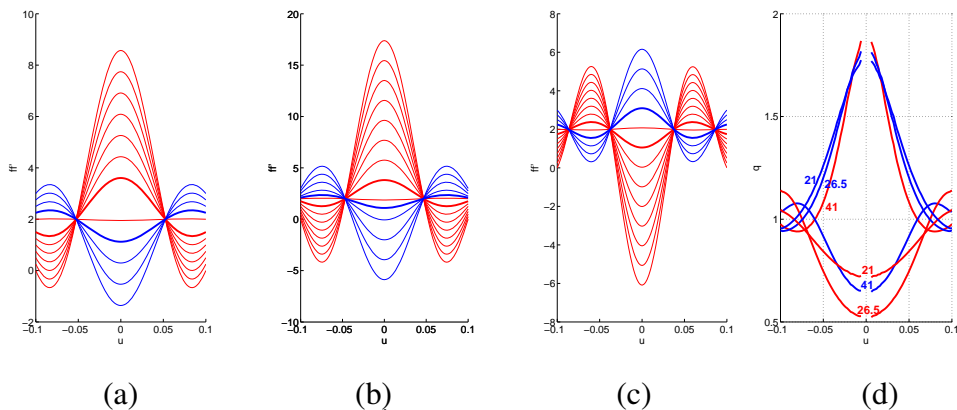


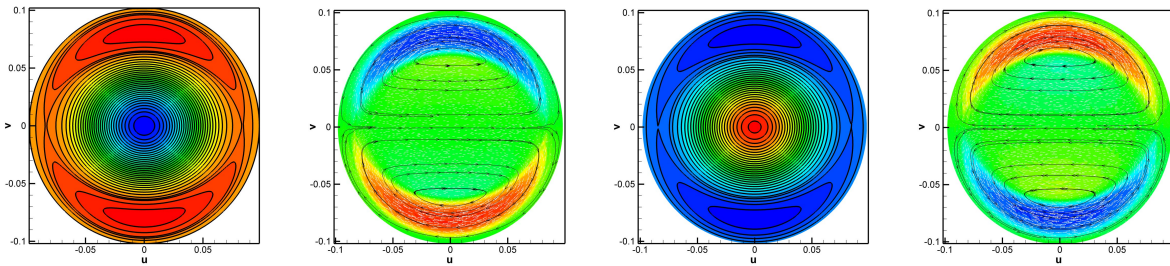
Figure 1. The current density $j_3 = ff'$ profiles versus u at $v = 0$ for the values of $0 \leq A \leq 2$ (red) and $2 < A \leq 3$ (blue) in the $R\kappa = 1$ equilibria (a) non-resonant value of $\alpha = 21$; $A = 1.75, 2.25$ are shown by bold lines (b) $\alpha = 26.5$ close to the $m = 2$ resonant value; $A = 1.75, 2.1$ are shown by bold lines (c) $\alpha = 41$ close to the $m = 3$ resonant value; $A = 1.75, 2.25$ are shown by bold lines (d) the safety factor q profiles for the values of A corresponding to the bold lines in current density plots, the numbers indicate α values.

2 Ideal MHD stability of helical equilibria with resonant and non-resonant islands

The stability computations were performed with the MHD_NX code [3] modified for an arbitrary 2D equilibrium configuration (cylindrical, toroidal axisymmetric and helical). The parallel direct sparse matrix solver MUMPS [4] was used within the PETSc framework running several times faster for large grid cases compared to the standard solver.

Let us note that the growth rate dependence on the value of $R\kappa$ under fixed ψ_h , ff' and q is quite weak in general due to the large aspect ratio approximation. So for the internal $n_h = 0$ mode stability calculations the value of $R\kappa = 1$ is used, which corresponds to islands at the $q = 1$ rational surface. Let us remind that any shaping of the plasma column, including spatial helical axis, is admissible also for $q = 1/n$ islands in general.

The results for the non-resonant islands in the equilibria with $\alpha = 21$ from [1] were extended to the negative shear (hollow current) configurations with $A > 2$. It was found that the growth rate of $n_h = 0$ tilt mode for the $m = 2$ island chain induced by elliptic cross-section deformation is almost the same in the negative shear equilibrium with the same global shear (q variation) (Fig.2). The triangular plasma shape deformation leads to marginally unstable $m = 3$ island chain; the equilibrium with squareness in the cross-section giving $m = 4$ islands is stable.



(a) (b) (c) (d)
Figure 2. (a) Level lines of helical flux ψ_h for the equilibrium with non-resonant $m = 2$ islands (cross-section elongation 1.04) $\alpha = 21$, $A = 2.25$, negative shear; (b) arrow plot and streamlines of plasma displacement ξ projection onto (u,v) plane and contour plot of ξ_z , $\omega/\omega_{Ah}^2 = -0.63$; (c) helical flux ψ_h for the equilibrium with non-resonant $m = 2$ islands $\alpha = 21$, $A = 1.75$, positive shear; (d) plasma displacement, $\omega/\omega_{Ah}^2 = -0.67$

The stability properties change for larger and resonant values of $\alpha \simeq 26.5, 41$. The same boundary perturbations with $m = 3$ for $\alpha = 26.5$ and with $m = 4$ for $\alpha = 41$ give $n_h = 0$ instability (Fig.3). Let us note that there are at least two unstable modes for the $m = 3$ and $m = 4$ island cases.

The resonant cases are more unstable than non-resonant cases for the same island width (Fig.4). Moreover, beside the main unstable mode (with multiplicity 1 for the $m = 2$ islands and with multiplicity 2 for the $m = 3$ islands) there are several other modes with lower growth rates. The growth rates in terms of helical Alfvén frequency $\omega_{Ah}^2 = (\psi_{h,max} - \psi_{h,min})^2 / (a^4 \rho)$ increase with the width of the islands (Table 1).

3 Discussion

The numerical stability calculations with the MHD_NX code show ideal MHD $n_h = 0$ tilt-type instability of the chains of magnetic islands in the equilibria with circular cross-section and resonant values of α . At the same time the islands generated by the plasma geometry perturbation in the non-resonant equilibria demonstrate destabilization with increasing α even for $m > 2$ geometry induced island chains. All that concerns the case of linear equilibrium solutions in large aspect ratio case with $f \approx \text{const}$. The helically symmetric equilibrium solutions that are relevant to RFP experiments are essentially nonlinear with non-constant longitudinal magnetic field function f . The investigation of the RFP equilibrium stability with islands and self-organized single magnetic axis equilibria [2] are subjects of future work.

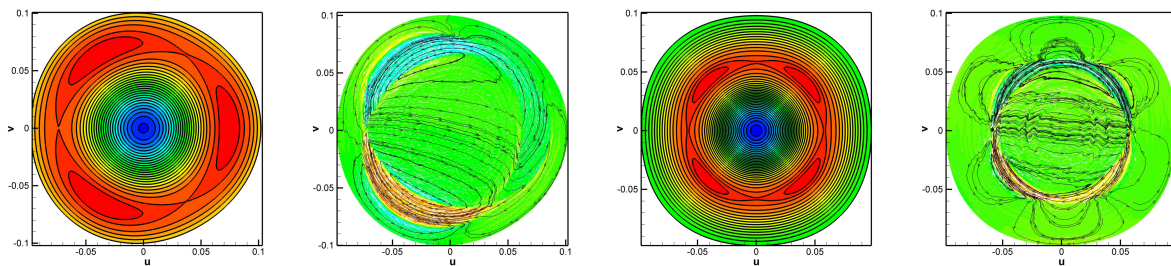


Figure 3. (a) Level lines of helical flux ψ_h for the equilibrium with non-resonant $m = 3$ islands $\alpha = 26.5$, $A = 2.1$; (b) arrow plot and streamlines of plasma displacement ξ projection onto (u, v) plane and contour plot of ξ_z , $\omega^2/\omega_{Ah}^2 = -0.15$; (c) helical flux ψ_h for the equilibrium with non-resonant $m = 4$ islands $\alpha = 41$, $A = 1.75$; (d) plasma displacement, $\omega^2/\omega_{Ah}^2 = -0.24$.

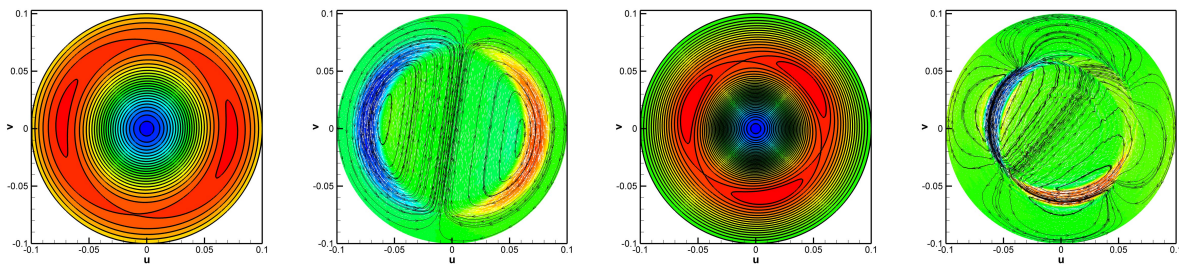


Figure 4. (a) Level lines of helical flux ψ_h for the equilibrium with resonant $m = 2$ islands $\alpha = 26.5$, $A = 2.1$, $H = 0.04$; (b) plasma displacement, $\omega^2/\omega_{Ah}^2 = -0.87$; (c) helical flux ψ_h for the equilibrium with resonant $m = 3$ islands $\alpha = 41$, $A = 1.75$, $H = 0.04$; (d) plasma displacement, $\omega^2/\omega_{Ah}^2 = -0.51$.

H for $\alpha = 26.5$, $A = 2.1$	0.01	0.02	0.04
$-\omega^2/\omega_{Ah}^2$	0.18	0.39	0.87
H for $\alpha = 41$, $A = 1.75$	0.01	0.02	0.04
$-\omega^2/\omega_{Ah}^2$	0.18	0.29	0.51

Table 1. Normalized squared growth rates $-\omega^2/\omega_{Ah}^2$ for two equilibria with circular cross-section and resonant helical islands vs the amplitude of the helical eigenfunction H .

- [1] S.Yu.Medvedev, A.A.Martynov, L.Villard, 39th EPS Conference and 16th Int. Congress on Plasma Phys., Stockholm, Sweden, 2 - 6 July, 2012 ECA Vol.36F, P1.082 (2012)
- [2] R. Lorenzini *et al.* Nature Physics **5**, 570 (2009)
- [3] S.Yu.Medvedev, A.A.Martynov, L.Villard, 38th EPS Conference on Plasma Phys. Strasbourg, France, 27 June - 1 July, 2011 ECA Vol.35G, PD2.07 (2011)
- [4] P. R. Amestoy, I. S. Duff and J.-Y. L'Excellent, Comput. Methods in Appl. Mech. Eng., **184**, 501-520 (2000)