

Energy approaches and dispersion relations for resistive wall modes

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1. Introduction. Stability of the resistive wall modes (RWMs) is often studied by using the dispersion relation

$$\gamma\tau_D = -W_\infty / W_b \quad (1)$$

proposed in [1] or its kinetic modifications [2]. Here the perturbed energies without and with ideal wall, W_∞ and W_b , are calculated as volume integrals over the whole space, including the plasma and outer regions. Accordingly, a plasma model is introduced from the very beginning to find W_∞ and W_b , which requires a knowledge of a full dynamics of the perturbed state. In [1], the plasma is treated as subject to the ideal MHD constraints.

Alternatively, the growth rate can be found from the energy balance outside the plasma [3, 4]. It follows from the Poynting theorem plus boundary conditions that

$$F = dW_{em}^{out}/dt + D, \quad (2)$$

where *out* denotes the space behind the wall inner surface S_w (the wall and outer vacuum),

$$D \equiv \int_{out} \mathbf{j} \cdot \mathbf{E} dV, \quad F \equiv \oint_{wall-} \mathbf{P} \cdot d\mathbf{S}_w = -d(W_{pl} + W_{em}^{gap})/dt, \quad (3)$$

W_{pl} , W_{em}^{gap} and W_{em}^{out} are, respectively, the plasma energy, magnetic energy in the plasma-wall vacuum gap and in the outer volume (in the RWM theory the plasma kinetic energy is disregarded), and $\mathbf{P} = \mathbf{E} \times \mathbf{B}$ with \mathbf{E} the electric field and $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ the magnetic induction.

When all perturbations vary in time as $\propto \exp(\gamma t)$, the dispersion relation is [3, 4]

$$\gamma D_w = 2(F_0 - W_{em}^{out}), \quad (4)$$

where $F_0 \equiv F/(2\gamma)$ and $D_w \equiv D/\gamma^2$. Efficiency of such approach was proved by incorporation of the thick-wall effects into analysis of the fast RWM [3, 4], while (1) is applicable in the thin-wall limit only (for slow RWMs).

Equations (1) and (4) are evidently different. Here we discuss this and other differences, inherent constraints of the models, and establish the relations between the approaches [1] and [3, 4].

2. Formulation of the problem. We consider the system: toroidal plasma – vacuum gap – resistive wall – outer vacuum region. Following [1], we represent the potentials φ_- and φ_+ of the magnetic perturbation $\mathbf{b} = \nabla\varphi$ in the plasma-wall gap and in the outer vacuum region as a superposition of two solutions without wall (or wall at infinity) and with ideal wall, φ_∞ and φ_b , each satisfying the Laplace equation:

$$\varphi_- = c_1\varphi_\infty + c_2\varphi_b, \quad \varphi_+ = c_3\varphi_\infty \quad (5)$$

with constant c_i and the boundary conditions on the plasma surface (\mathbf{n}_{pl} is the unit normal)

$$\mathbf{n}_{pl} \cdot \nabla\varphi_\infty = \mathbf{n}_{pl} \cdot \nabla\varphi_b = \mathbf{n}_{pl} \cdot \nabla\varphi_- \quad (6)$$

and $\mathbf{n}_w \cdot \nabla\varphi_b = 0$ at the wall (\mathbf{n}_w is the unit normal to S_w). These conditions are satisfied at

$$c_1 + c_2 = 1, \quad c_1 = c_3, \quad (7)$$

if the wall is assumed magnetically thin as in [1]: $\mathbf{n}_w \cdot \nabla\varphi_- = \mathbf{n}_w \cdot \nabla\varphi_+$. Now we use (5)–(7) to transform (4) so that we could compare it with (1).

3. Transformation of (4). In the thin-wall approximation, \mathbf{E} must be almost constant across the wall. Then for a wall with uniform thickness d we find from the Ohm's law in the simple form $\mathbf{j} = \sigma\mathbf{E}$ with $\sigma = \text{const}$ (the same model as used in [1]):

$$\gamma^2 D_w \equiv \int_{wall} \mathbf{j} \cdot \mathbf{E} dV = \sigma d \oint_{wall} \mathbf{E}^2 dS_w. \quad (8)$$

On the right hand side of (4) we have [4]

$$2F_0 \equiv F/\gamma = - \oint_{wall-} \varphi_- \mathbf{b} \cdot d\mathbf{S}_w, \quad 2W_{em}^{out} \equiv \int_{out} \mathbf{b}^2 dV = - \oint_{wall+} \varphi_+ \mathbf{b} \cdot d\mathbf{S}_w, \quad (9)$$

where F_0 was transformed by using $\nabla \times \mathbf{E} = -\partial\mathbf{b}/\partial t$ and the equality

$$\mathbf{E} \times \nabla f = -\nabla \times f\mathbf{E} + f\nabla \times \mathbf{E}, \quad (10)$$

integration of which over a closed surface nullifies the first term. Then with $\mathbf{b}_\infty = \nabla\varphi_\infty$ and a consequence of (5) that $\varphi_+ - \varphi_- = -c_2\varphi_b$ at the wall, we obtain

$$2(F_0 - W_{em}^{out}) = \oint_{wall+} \varphi_+ \mathbf{b} \cdot d\mathbf{S}_w - \oint_{wall-} \varphi_- \mathbf{b} \cdot d\mathbf{S}_w = - \oint_{wall} c_1 c_2 \varphi_b \mathbf{b}_\infty \cdot d\mathbf{S}_w. \quad (11)$$

Here Eq. (7) and the boundary condition $\mathbf{n}_w \cdot \nabla\varphi_b = 0$ have been used. In the thin-wall approximation, the integrals in (11) are evaluated on the same surface denoted *wall*.

Substituting (8) and (11) into (4) we obtain

$$\gamma \alpha \bar{b} = c_2 / c_1, \quad (12)$$

where \bar{b} is a measure of the average radius of the vacuum chamber introduced by Eq. (68) in [1]. In our notation with \mathbf{E}/c_1 exactly the same as $-\gamma\delta\mathbf{A}_\infty$ in [1] we have

$$\bar{b} \oint_{wall} \varphi_b \mathbf{b}_\infty \cdot d\mathbf{S}_w = - \oint_{wall} \frac{\mathbf{E}^2}{\gamma^2 c_1^2} dS_w. \quad (13)$$

4. Calculation of c_2/c_1 in terms of energies. With (1) as a target for comparison we have to express c_2/c_1 in (12) via W_∞ and W_b related to the plasma energy W_{pl} by

$$F_0^{pl} - W_v^{(\infty)} = -(W_v^{(\infty)} + W_{pl}) = -W_\infty, \quad (14)$$

$$F_0^{pl} - W_v^{(b)} = -(W_v^{(b)} + W_{pl}) = -W_b. \quad (15)$$

Here $F_0^{pl} \equiv F^{pl}/(2\gamma)$ with F^{pl} the energy flux through the plasma surface,

$$F^{pl} \equiv \oint_{plasma} \mathbf{P} \cdot d\mathbf{S}_{pl} = -dW_{pl}/dt, \quad (16)$$

which is similar to the flux F through the wall in (3), and

$$2W_v^{(\infty)} = - \oint_{plasma} \varphi_\infty \mathbf{b}_\infty \cdot d\mathbf{S}_{pl}, \quad 2W_v^{(b)} = - \oint_{plasma} \varphi_b \mathbf{b}_b \cdot d\mathbf{S}_{pl} \quad (17)$$

define the magnetic energies $W_v^{(\infty)}$ and $W_v^{(b)}$ outside the plasma for solutions $\mathbf{b}_\infty = \nabla\varphi_\infty$ and $\mathbf{b}_b = \nabla\varphi_b$ with boundary conditions (6). To transform the left hand sides of (14) and (15) we use the equalities

$$F^{pl} - \frac{d}{dt} \int_{vac} \frac{\mathbf{b}_\alpha^2}{2} dV = - \oint_{plasma} (\varphi - \varphi_\alpha) \frac{\partial \mathbf{b}}{\partial t} \cdot d\mathbf{S}_{pl}, \quad (18)$$

where $\mathbf{b}_\alpha = \nabla\varphi_\alpha$ and $\alpha = \infty$ or b . Then with $\varphi - \varphi_\alpha = c_1\varphi_\infty + c_2\varphi_b - (c_1 + c_2)\varphi_\alpha$ we obtain

$$c_2/c_1 = -W_\infty/W_b, \quad (19)$$

so that (12) turns into

$$\gamma\alpha\bar{b} = -W_\infty/W_b. \quad (20)$$

Finally, with “resistive diffusion time of the wall” $\tau_D = \alpha\bar{b}$ defined by Eq. (66) in [1] we convert the dispersion relation (20) into the form (1). That was the main goal of our analysis.

5. Cylindrical estimate of τ_D . In RWM studies the “wall time” $\tau_w = \alpha dr_w$ is often used instead of τ_D . To find their relation we consider a plasma cylinder surrounded by a coaxial resistive wall and a single mode with $m > 0$, see [4]. In this case, we have for the radial component $b_\alpha^r = \text{Re}[b_\alpha(r, t) \exp(im\theta - in\zeta)]$ of low- m magnetic perturbations

$$b_b = g(x^{-m-1} - x^{m-1}), \quad b_\infty = hx^{-m-1}, \quad (21)$$

where $x = r/r_w$. It follows from (6) that $g(x_{pl}^{-m-1} - x_{pl}^{m-1}) = hx_{pl}^{-m-1}$. Then $b_b^{out} = -b_\infty/(1 - x_{pl}^{2m})$, which is the contribution to b_b from wall, and, approximately,

$$\varphi_\alpha = (r/m)(b_\alpha^{out} - b_\alpha^{in}) = (r/m)(2b_\alpha^{out} - b_\alpha) \quad (22)$$

with $\mathbf{b}_\alpha = \nabla \varphi_\alpha$. Now the integral on the left hand side of (13) turns into

$$\oint_{wall} \varphi_b \mathbf{b}_\infty \cdot d\mathbf{S}_w = (2r_w/m) \oint_{wall} b_b^{out} b_\infty dS_w = -\frac{2r_w}{m(1-x_{pl}^{2m})} \oint_{wall} b_\infty^2 dS_w. \quad (23)$$

To transform the right hand side of (13), from $\nabla \times \mathbf{E} = -\partial \mathbf{b} / \partial t$ in cylindrical geometry we have $\partial E_\zeta / \partial \theta - (r_w/R_0) \partial E_\theta / \partial \zeta = -\gamma r_w b_\alpha^r$. For systems with $r_w/R_0 \ll 1$ this yields

$$E_\zeta \approx \gamma \frac{r_w}{m^2} \frac{\partial b_\alpha^r}{\partial \theta}. \quad (24)$$

Then with (24) and $\mathbf{E}^2 \approx E_\zeta^2$ the right hand side of (13) for $\mathbf{E} = c_1 \mathbf{E}_\infty$ can be written as

$$- \oint_{wall} \frac{\mathbf{E}^2}{\gamma^2 c_1^2} dS_w = -\frac{r_w^2}{m^2} \oint_{wall} b_\infty^2 dS_w. \quad (25)$$

Substituting (23) and (25) into (13) we obtain

$$\bar{b} = r_w(1 - x_p^{2m})/2m, \quad \tau_D = \tau_w(1 - x_p^{2m})/2m. \quad (26)$$

Such cylindrical τ_D is often used in numerical calculations even for toroidal systems [5–7].

6. Discussion. Here we have proved that dispersion relation (1) proposed in [1] can be obtained from (4) within the approach [3, 4] if the constraints of [1] are imposed. These are expressed by equations (5), which can be justified in the cylindrical model, and thin-wall boundary conditions $c_1 = c_3$ with \mathbf{E} assumed constant across the wall, see Eqs. (45) and (56) in [1]. Precisely, Eqs. (1) and (4) are equivalent for slow RMWs (magnetically thin resistive wall) and the ideal plasma. Eq. (4), however, is valid in a wider area in γ and for nonideal plasma too, where Eq. (1) cannot be used. The analysis here is performed for real γ .

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