Continuum kinetic model for low-collisionality plasmas using adaptive mesh refinement

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Abstract

Recent advancements in supercomputing capability make it possible to solve the Maxwell-Boltzmann equation system in full six-dimensional phase space. While computationally intensive, continuum models such as these present a viable alternative to particle-in-cell methods because they can be cast in conservation-law form, are not susceptible to sampling noise, and can be implemented using high-order numerical methods. A fourth-order accurate method has been developed to solve the continuum kinetic Vlasov-Poisson model in one spatial and one velocity dimension. The governing equation is solved in its conservation-law form using a finite volume representation. Adaptive mesh refinement (AMR) is used to allow for efficient use of computational resources while maintaining desired levels of resolution. The model is tested on several plasma phenomena including: weak and strong Landau damping and the two-stream instability.

Introduction

Due to recent advancements in supercomputing technology, full phase-space continuum methods have become a viable means of simulating nonlinear plasma kinetics. [1, 2, 3, 4, 6] Continuum methods are advantageous because they can be cast in conservation-law form and allow for: straightforward parallelization based on domain decomposition; the use of adaptive mesh refinement (AMR) techniques; and numerical methods that are high-order accurate in space and time.[2] Parallel AMR can, in particular, be used to reduce the cost of a continuum multi-dimensional Vlasov simulation by focusing computational resources in dynamic parts of the domain. Moreover, continuum methods do not suffer from sampling-associated noise seen in PIC methods [9] and can thus produce more physically accurate results.

The content of this paper investigates a fourth-order accurate numerical method for modeling the electrostatic regime represented by the Vlasov-Poisson system in one spatial and one velocity dimension. The unsplit finite volume method is fourth-order accurate in time and space and solves the system of equations in conservation-law form. The method is benchmarked against analytic results for weak and strong Landau damping and the two-stream instability. Results with adaptive mesh refinement are presented.

Vlasov-Poisson System and Underlying Assumptions

Plasma evolution timescales over which electrons are dynamic and ions remain static are considered. In such a system there is only one evolving distribution function $f(x,v,t)$ — that

of the electrons. Assuming collisions have negligible effect means the plasma kinetics can be modeled by the 1D Vlasov equation [7], which in conservation-law form is

$$0 = \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(vf) - \frac{e}{m}\frac{\partial}{\partial v}(Ef), \tag{1}$$

where e is the absolute value of the electron charge. In Eq. 1 only electrostatic forces are considered such that currents and magnetic fields are assumed to be negligible. The electric potential, ϕ, is calculated using Poisson's equation in one dimension, $\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho_c}{\varepsilon_0}$. Note that the electric field is related to the gradient of the potential, $E = -\nabla\phi$, and the charge density is defined as

$$\rho_c(x) = e\left[1 - \int_{-\infty}^{\infty} f(x,v)dv\right]. \tag{2}$$

The distribution function $f(x,v)$ is normalized such that the net charge density over the entire spatial domain is zero.

Discretization and Integration in Time

A fourth-order finite volume method is employed to advance the solution $f(x,v)$ in time. The algorithm is summarized below:

1. Initialize distribution function by fourth-order cell-average, denoted by $\langle\cdot\rangle$:

$$\langle f\rangle_{i,j} = f^0(ih_x, jh_v) + \frac{1}{24}\left(f^0_{i+1,j} - 2f^0_{i,j} + f^0_{i-1,j}\right) + \frac{1}{24}\left(f^0_{i,j+1} - 2f^0_{i,j} + f^0_{i,j-1}\right). \tag{3}$$

2. Solve Poisson equation using fourth-order finite volume stencil. [8]

3. Compute cell-average electric field from the potential,

$$\langle E\rangle_i = -\frac{8}{12h_x}\left(\langle\phi\rangle_{i+1} - \langle\phi\rangle_{i-1}\right) + \frac{1}{12h_x}\left(\langle\phi\rangle_{i+2} - \langle\phi\rangle_{i-2}\right). \tag{4}$$

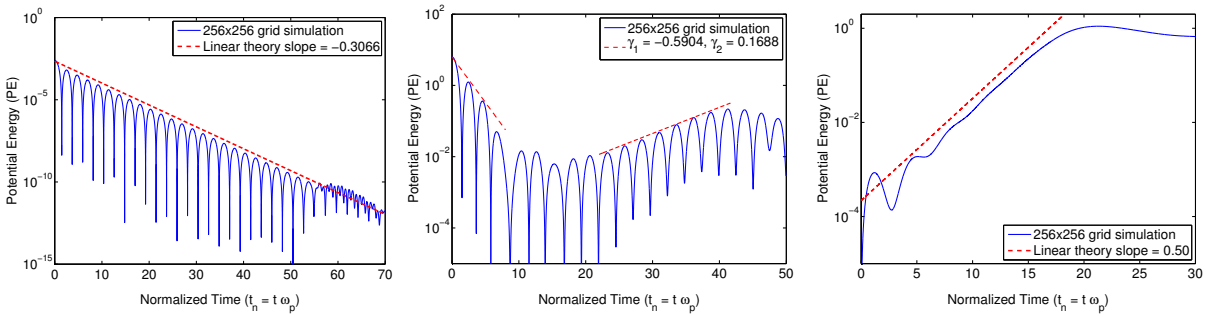
4. Compute fluxes and advance solution using fourth-order Runge-Kutta

$$\frac{d\langle f\rangle_{i,j}}{dt} = -\frac{1}{h_x}\left[\langle fv\rangle_{i+\frac{1}{2},j} - \langle fv\rangle_{i-\frac{1}{2},j}\right] + \frac{e}{m}\frac{1}{h_v}\left[\langle fE\rangle_{i,j+\frac{1}{2}} - \langle fE\rangle_{i,j-\frac{1}{2}}\right]. \tag{5}$$

Single-Grid Simulation Results

The single grid algorithm described in the previous section is benchmarked against analytic results from kinetic theory. A standard test case is to simulate weak Landau damping by initializing a Maxwellian distribution in velocity space with a small position-dependent perturbation:

$$f(x,v)|_{t=0} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)(1 + a\cos(kx)), \tag{6}$$



(a) Weak Landau damping (b) Strong Landau damping (c) Two-stream

Figure 1: Simulation results for (a) weak Landau damping, (b) strong Landau damping, and (c) two-stream instability using a 256×256 grid. The evolution of potential energy demonstrates agreement with theoretical predictions.

with $k = 0.5$ and $a = 0.01$. In weak Landau damping, potential energy is transformed into kinetic energy as indicated by a steady net decrease in the value of the former, shown in Fig. 1(a). In strong Landau damping, the Maxwellian in Eq. 6 is given a large perturbation with amplitude $a = 0.5$. In this case, the potential energy in the system evolves non-linearly. This evolution is shown in Fig. 1(b). In both cases, the decay and growth rates in potential energy from the simulation are shown to be in good agreement with theoretical predictions.

The algorithm is also used to simulate the two-stream instability. This simulation is initialized using the following distribution function

$$f(x,v)|_{t=0} = \frac{1}{\sqrt{2\pi}} v^2 \exp\left(-\frac{v^2}{2}\right)(1 + a\cos(kx)) \tag{7}$$

with $k = 0.5$ and $a = 0.01$. The spatial perturbation results in inhomogeneities in the electron distribution that cause kinetic energy to be converted into potential energy. The growth rate of the potential energy is consistent with the theoretically predicted value for the two-stream instability, as shown in Fig. 1(c).

Two-stream Instability with Adaptive Mesh Refinement

Adaptive mesh refinement is implemented into the single-grid algorithm described above, using the techniques described in Ref. [5]. As indicated in Eq. 2, the charge density is computed by taking a velocity moment of the distribution function at each spatial location. To do this on multiple levels of grids requires that additional interpolation steps be added to the AMR algorithm. In particular, when computing the zeroth moment of $f(x,v)$, the integral with respect to velocity is performed using the finest grid information available in a given part of the domain.

As a demonstration of adaptive mesh refinement capability, the two-stream instability is simulated using a three-level adaptively refined grid. The evolution of the distribution function and grid is shown in Fig. 2. The simulation uses a base grid of size 64×64 with a factor of four refinement for each successive level. Further analysis is required to evaluate the merits of AMR

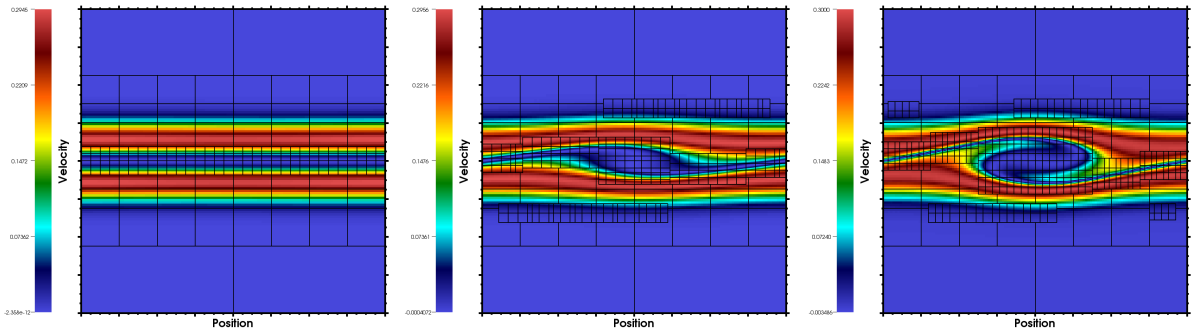


Figure 2: Two-stream instability at $t = 0.2, 18.1, 22.1$. AMR simulation with 64×64 base grid, three levels of grid refinement, and a refinement ratio of four. Thus the smallest boxes indicate regions of the domain that have sixteen times the resolution of the coarsest grid.

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and to determine to what extent it reduces the computational cost of Vlasov-Poisson simulations.

Conclusions

A fourth order accurate algorithm in space and time has been developed to solve the Vlasov-Poisson system in one spatial and one velocity dimension. The simulation results demonstrate close agreement with theoretical predictions, as tested on weak Landau damping, strong Landau damping, and the two-stream instability. AMR is also demonstrated as a means of reducing computational cost by focusing computational resources in dynamic regions of the domain. Further work is needed to assess the speed-up offered by adaptive mesh refinement, specifically as it applies in multiple dimensions. The described algorithm will also benefit from the use of limiters, which will address dispersive oscillations and lack of positivity preservation that are inherent to high-order finite volume calculations.

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