CALCULATION OF LIAPUNOV LENGTHS FOR THE SIMPLE MAP FOR DIVERTOR TOKAMAKS

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ABSTRACT: The fraction f_{20} of field lines starting inside the last good surfaces for which the ratio $l/l_L > 20$ before striking the collector plate in divertor tokamaks is calculated as a function of magnetic asymmetries and radial diffusion of guiding centers. l and l_L are the length travelled by the field line and the Liapunov length corresponding to the largest Liapunov exponent for the field line. Magnetic topology of divertor tokamaks is represented by the simple map [1]. Simple map is the simplest symplectic map that has the generic magnetic topology of divertor tokamaks. Radial diffusion of particles is represented by the radial expansion coefficient D. Magnetic asymmetries are represented by map parameter k, as in the standard map. f_{20} is calculated as a function of k and D. For magnetic perturbations of 10E-3 or higher and D around 10E-6, f_{20}=1. This is in agreement with Boozer prediction [2]. Magnetic perturbations of 10E-3 and radial diffusion of 10E-6 can be critical for plasma confinement. This work is supported by the US DOE grants DE-FG02-01ER54624 and DE-FG02-04ER54793. This research used resources of the NERSC, supported by the Office of Science, US DOE, under contract DE-AC02-05CH11231.

Liapunov lengths for field lines starting on good surfaces inside the last good surfaces are calculated as function of perturbation amplitude and diffusion in the simple map [1]. Size of magnetic perturbation $\delta=\delta B/B$ is represented by map parameter k. δ is calculated from the ratio of the root mean square deviation of the Hamiltonian on the last good surface from 10^6 iterations of the map to the equilibrium Hamiltonian on the surface. Data used is summarized in Table I.

Table I. The map parameter k, corresponding size of magnetic perturbation δ, the width w of stochastic layer near the X-point, the last good surface y_{LGS}, and the number of iterations N_p of the simple map that is equivalent to a single toroidal circuit of the tokamak.

k	$\delta=\delta B/B$	$w(k)$ (m)	y_{LGS} (m)	N_p
0.55	$1.0309\text{X}10^{-6}$	$9.35\text{X}10^{-4}$	0.999065	11

0.597	1.0399X10^{-5}	2.687X10^{-3}	0.997313	10
0.646	1.1801X10^{-4}	8.245X10^{-3}	0.991755	9
0.719	1.0506X10^{-3}	2.178X10^{-2}	0.978223	8
0.801	1.4666X10^{-2}	7.5858X10^{-2}	0.924142	7

The simple map can be written as $\overrightarrow{X}_{n+1} = \overleftrightarrow{T}_n \overrightarrow{X}_n$, where $\overrightarrow{X}_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$, $\overrightarrow{X}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, $\overleftrightarrow{T}_n = \begin{bmatrix} 1 & -k(1-y_n) \\ k & 1-k^2(1-y_n) \end{bmatrix}$, and $\left|\overleftrightarrow{T}_n\right| = +1$. We introduce effects of particle diffusion by applying the radial expansion operator, $x_i \rightarrow \left(1 + r_i D\right) x_i$, where D is radial expansion coefficient, r_i is a random number distributed uniformly between 0 and 1. Radial displacement operator is represented by $\overrightarrow{X}_n \rightarrow \overleftrightarrow{D}_n \overrightarrow{X}_n$, where $\overleftrightarrow{D}_n = \begin{bmatrix} 1 + r_n D & 0 \\ 0 & 1 \end{bmatrix}$. Operator $\overleftrightarrow{D}_n$ is applied each time a field line crosses the principal plane of the tokamak. The differential of the simple map is $\overrightarrow{dX}_{n+1} = \overleftrightarrow{J}_n \overrightarrow{dX}_n$, where $\overrightarrow{dX}_{n+1} = \begin{bmatrix} dx_{n+1} \\ dy_{n+1} \end{bmatrix}$, $\overrightarrow{dX}_n = \begin{bmatrix} dx_n \\ dy_n \end{bmatrix}$, and $\overleftrightarrow{J}_n = \begin{bmatrix} 1 & -k\left(1-2y_n\right) \\ k & 1-k^2\left(1-2y_n\right) \end{bmatrix}$. $\overleftrightarrow{J}_n$ is the Jacobian of transformation, and $\left|\overleftrightarrow{J}_n\right| = +1$. s_0 is the initial separation between a given field line and a neighboring line. The separation is $s_{n+1}(l)$ after the field line travels a distance l. In the limit as $s_0 \rightarrow 0$, the separation, including the radial displacement represented by the operator D, is calculated from $\overrightarrow{dX}_{n+1} = \overleftrightarrow{L}_n \overrightarrow{dX}_n$, where $\overleftrightarrow{L}_n = \overleftrightarrow{D}_n \overleftrightarrow{J}_n$. We write $\overrightarrow{dX}_{n+1}$ as $\overrightarrow{dX}_{n+1} = \left(\overleftrightarrow{L}_n \overleftrightarrow{L}_{n-1} ... \overleftrightarrow{L}_1 \overleftrightarrow{L}_0\right) \overrightarrow{dX}_0 = \overleftrightarrow{\mathcal{L}}\, \overrightarrow{dX}_0$. The eigenvalues of $\overleftrightarrow{\mathcal{L}}$ are $\frac{1}{2}\left[trace\left(\overleftrightarrow{\mathcal{L}}\right) \pm \sqrt{\left(trace\left(\overleftrightarrow{\mathcal{L}}\right)\right)^2 - 4det\left(\overleftrightarrow{\mathcal{L}}\right)}\right]$. The separation is then given by $s_{n+1}(l) = \sqrt{\left(\overrightarrow{dX}_{n+1}\right)^{\dagger} \overrightarrow{dX}_{n+1}} = \sqrt{\left(\overrightarrow{dX}_0\right)^{\dagger} \left(\overleftrightarrow{\mathcal{L}}^{\dagger} \overleftrightarrow{\mathcal{L}}\right) \overrightarrow{dX}_0}$. Continuous analog of the simple map is used to calculate analytically the length l_{n+1} of the field line between the n^{th} to $(n+1)^{th}$ iteration of the simple map. The total length of field line up to the end of $(n+1)^{th}$ iteration is $\ell = \sum_{1}^{n+1} \ell_j + \sum_{1}^{mod(n+1, N_p)} \ell_{D,j}$. $l_{D,j}$ is the distance traveled by the line when the radial diffusion operator is applied in the principal plane. The largest positive eigenvalues λ of the matrix $\overleftrightarrow{\mathcal{L}}$

corresponding to the largest separation given from the tangent map is calculated from Grasso *et al* expression [3] $max \left\| \delta \vec{x}(z) \right\| = e^{\sigma(z,z_0,\vec{x}_0)\,|z-z_0|} \left\| \delta \vec{x}(z_0) \right\|$, with $\sigma(z,z_0,\vec{x}_0)$ given by $\sigma(z,z_0,\vec{x}_0) = \frac{1}{|z-z_0|} ln\sqrt{\lambda\left(z,z_0,\vec{x}_0\right)}$. Here $z = l$, $z_0 = 0$, $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $l_L = 1/\sigma(l,\vec{x}_0)$ l_L is the Liapunov length [2] for the field line, $\vec{x}_0$ is the starting position of the field line, λ is the magnitude of the largest real eigenvalue of the matrix $\vec{\mathcal{L}}$.

For each k, we construct seven to eight good surfaces lying between the y_{LGS} and (y_{LGS}-0.1). y_{LGS} is where the last good surface intersects the y-axis near the X-point. Each surface is made of 10,000 points. These 70,000 or 80,000 lines for each k are advanced for at most 10,000 toroidal circuits of tokamak. Every time a line is in the principal plane of tokamak, the radial diffusion operator is applied. After every iteration of map, eigenvalues of $\vec{\mathcal{L}}$ for the line are calculated, and if the eigenvalues are real, one with larger magnitude is chosen. From the length of the line and the larger eigenvalue, the Lyapunov length l_L for the line is calculated. Number of lines that have $l/l_L > 20$ before striking the collector plate at y=1 or before completing 10,000 circuits is calculated as function of the perturbation strength δ and diffusion D. D values chosen are 10^{-9}, $2X10^{-9}$, $3X10^{-9}$, … , $8X10^{-3}$, $9X10^{-3}$, and 10^{-2}. Fractal dimension of footprints $d_f(\delta,D)$, areas of footprints $A(\delta,D)$, distributions of lengths $l(\delta,D)$, distributions of Lyapunov lengths $l_L(\delta,D)$, averages of Lyapunov lengths $< l_L>(\delta,D)$, and other parameters and averages are calculated.

$f_{20}(\delta,D)$ is the fraction of lines starting inside the last good surface for which the ratio $l/l_L > 20$. For δ=1.04X10^{-5}, f_{20} is nonzero when $3X10^{-9} \leq D \leq 10^{-7}$, and has two maxima of 0.46 and 0.36 at D=5X10^{-8} and 6X10^{-7}, respectively. For δ=1.18X10^{-4}, f_{20} is nonzero when $3X10^{-8} \leq D \leq 6X10^{-6}$, and has two very closely placed almost equal maxima of 0.50 and 0.51 at D=3X10^{-7} and 6X10^{-7}, respectively. When the perturbation amplitude is raised to δ=1.05X10^{-3}, a large change sets in the size and profile of $f_{20}(D)$. The range in D for which $f_{20} \neq 0$ becomes considerably wider; $f_{20} \neq 0$ for $5X10^{-9} \leq D \leq 4X10^{-5}$, and has a single maximum of 0.98 at D=9X10^{-7}. The width of the domain in D for which f_{20} is nonzero when δ = 1.05X10^{-3} is roughly two orders of magnitude larger than the cases when δ =1.04X10^{-5} and 1.18X10^{-4}, and secondly the height of maxima of f_{20} is roughly twice that in the cases of smaller perturbation amplitudes. The maximum value of f_{20} now almost reaches unity. The combination $\delta \cong 10^{-3}$ and $D \cong 10^{-6}$ can be undesriable for plasma confinement. As the perturbation amplitude is further raised to δ=1.47X10^{-2}, this changed behavior of f_{20}

continues. Now range in D for which $f_{20} \neq 0$ becomes even wider; $f_{20} \neq 0$ for $10^{-9} \leq D \leq 2\text{X}10^{-4}$, and has a single maximum of 1.00 at D=3X10^{-6}. From these results, we conclude that the amplitude $\delta=10^{-3}$ is a critical value for tokamak fusion plasma confinement, as predicted by Boozer [2]. The results on $f_{20}(\delta,D)$ are shown in Fig. 1.

Average Liapunov length $<l_L>$ scales roughly as negative half power of D. See Fig. 2. The exact power law dependence is: For δ = 1.47X10^{-2}, $<L_L>$ scales as $D^{-0.54}$ with R^2 = 0.98; for δ = 1.05X10^{-3}, $<L_L>$ scales as $D^{-0.56}$ with R^2 = 0.98; for δ = 1.18X10^{-4}, $<L_L>$ scales as $D^{-0.58}$ with R^2 = 0.98; and for δ = 1.04X10^{-5}, $<L_L>$ scales as $D^{-0.57}$ with R^2 = 0.98. R^2 is the goodness of fit for the least square power fit to the data.

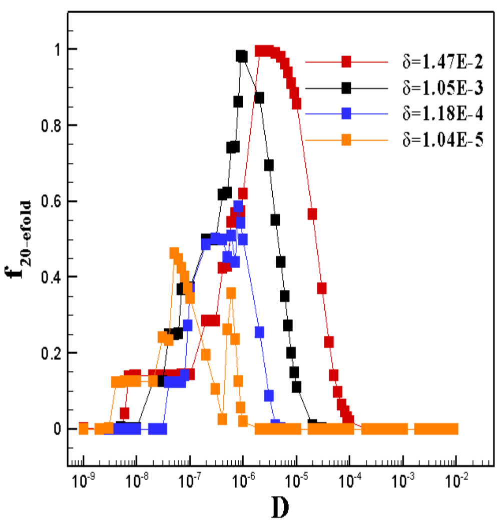

Fig. 1. The fraction f_{20} of field lines starting inside the last good surfaces in the simple map for which the ratio $l/l_L > 20$ before striking the collector plate as a function of the perturbation amplitude δ and radial expansion coefficient D. l and l_L are the length travelled by the field line and the Liapunov length corresponding to the largest Liapunov exponent for the field line.

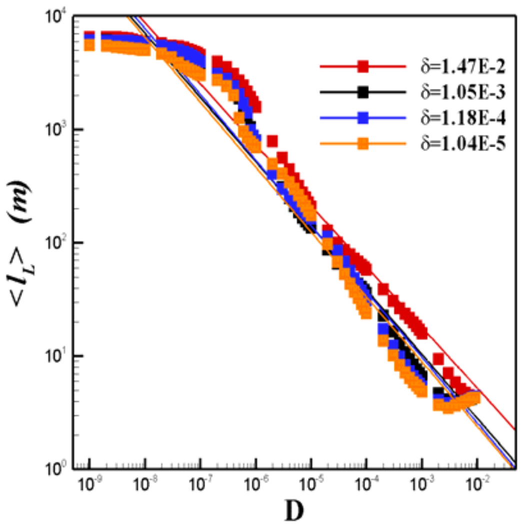

Fig. 2. The average Liapunov length $\langle l_L \rangle$ as a function of radial expansion coefficient D. $\langle l_L \rangle$scales roughly as $1/\sqrt{(D)}$. The solid lines are least square power fits.

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