Three-dimensional peeling-ballooning theory in magnetic fusion devices

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Abstract

The ideal linear MHD stability of general 3D magnetic configurations is thought to be important for topics such as the understanding and control of Edge Localized Modes (ELMS), the limit of the size of the pedestal that governs the high confinement H-mode, etc. Of these, the most important instabilities are assumed to be the so-called intermediate to high *n* peeling-ballooning modes, which combine high *n* ballooning modes with a lower *n* peeling nature. In this work, a first, general 3D theory is derived, with the result being a set of coupled second order ordinary differential equations, with appropriate boundary conditions, that reflect the minimization of the total energy of a plasma-vacuum system. This theory can be used to investigate 3D effects on (edge) plasma stability in tokamaks where the axisymmetry is broken by a toroidal ripple, a tritium breeder module, etc, as well as the stability of full 3D configurations such as stellerators.

Introduction

Linear ideal MHD stability can be a rather idealized approximation of reality, but it can also be surprisingly relevant, as the instabilities due to MHD activity often set important limitations on the performance of magnetic configurations. Therefore, it has been extensively used in the past and often gives a surprisingly good approximation of reality.

Of interest has been the treatment of so-called intermediate to high *n peeling-ballooning modes* in general 3D configurations. *High n* refers to a mode that is highly localized around a specific magnetic field line, which is the case for ballooning modes, for example. Peeling modes, on the other hand, can be thought of as an edge-localized version [5] of an interchange mode, and they are typically less localized poloidally and toroidally (though indeed radially), as indicated by the term *intermediate n* mode.

Coupled peeling-ballooning modes have been identified as probable protagonists in the behavior of Edge Localized Modes (ELMS), as well as the processes that limit the size of the pedestal that governs the high confinement H-mode [2, 8]. It is therefore very important that they are thoroughly understood and that they can be modelled accurately.

A missing gap in this theoretical description has been the fast and reliable treatment in general 3D configurations. On the one hand a numerical code, called ELITE, exists, which is based on theory developed originally by Connor [1] for ballooning modes and later extended to include peeling modes as well, by retaining terms of higher order in a high *n* ordering scheme [2]. The main limitation is the fact that ELITE works in 2D. For 3D there are various options, such as MISHKA [6] and KINX [3], which are indeed successful at describing various phenomena, but their downside is that they can be slower. To patch this hole, the authors are working on a new numerical code called PB3D (PEELING BALLOONING in 3D), which retains the high *n* ordering, including higher orders, just like ELITE, but applying to 3D configurations as well as 2D ones.

The first step in this work was the derivation of a suitable stability theory built on ideal linear MHD [7]. A discussion concerning the 3D theory will be the topic of the next section. Currently, work is being done on the new numerical code PB3D and a preliminary version is nearly ready. Some numerical issues will be discussed in the following section. Finally, in the near future, this code will first be benchmarked with other codes, and then compared with experiments.

Ideal linear 3D MHD stability

In ideal linear MHD stability of a plasma-vacuum system, the plasma energy is perturbed by $\boldsymbol{\xi}$ and the vacuum magnetic field by $\mathbf{Q}$. The stationary values of the *Rayleigh quotient* Λ are the Eigenvalues of the system and their signs determine whether this perturbation is stable or unstable:

$$\Lambda\left[\boldsymbol{\xi},\mathbf{Q}_\mathrm{v}\right] \equiv \frac{W\left[\boldsymbol{\xi},\mathbf{Q}_\mathrm{v}\right]}{I\left[\boldsymbol{\xi}\right]} \equiv \frac{W_\mathrm{p}\left[\boldsymbol{\xi}\right] + W_\mathrm{s}\left[\xi_n\right] + W_\mathrm{v}\left[\mathbf{Q}_\mathrm{v}\right]}{\frac{1}{2}\int_\mathrm{p} \rho \left|\boldsymbol{\xi}\right|^2 \mathrm{d}\mathbf{r}} \ . \tag{1}$$

where W_p, W_s and W_v are the contribution to the potential energy due to the plasma, the plasma-vacuum surface and the vacuum [4].

The three components of the perturbed potential energy can be calculated and the result is given by:

$$\begin{cases} \delta W_F\left(\boldsymbol{\xi}\right) = \dfrac{1}{2}\displaystyle\int_\mathrm{p} \mathrm{d}\mathbf{r}\left[\dfrac{|\mathbf{Q}|^2}{\mu_0} - \boldsymbol{\xi}^*\cdot\mathbf{j}\times\mathbf{Q} + \gamma p\left|\nabla\cdot\boldsymbol{\xi}\right|^2 + \left(\boldsymbol{\xi}\cdot\nabla p\right)\nabla\cdot\boldsymbol{\xi}^*\right] \\ \delta W_\mathrm{s}\left(\xi_n\right) = \dfrac{1}{2}\displaystyle\int_\mathrm{s} \mathrm{d}S\left[\left|\mathbf{n}\cdot\boldsymbol{\xi}\right|^2 \mathbf{n}\cdot\left[\!\left[\nabla\left(\mu_0 p + \dfrac{B^2}{2}\right)\right]\!\right]\right]_\mathrm{s} \\ \delta W_\mathrm{v}\left(\mathbf{Q}_\mathrm{v}\right) = \dfrac{1}{2}\displaystyle\int_\mathrm{v} \mathrm{d}\mathbf{r}\left[\dfrac{|\mathbf{Q}_\mathrm{v}|^2}{\mu_0}\right] , \end{cases} \tag{2}$$

where all the symbols have their usual meaning. Valid perturbations have to satisfy the *essential boundary conditions*:

$$\begin{cases} \boldsymbol{\xi} \text{ regular} & \text{(on } V\text{)} \\ \mathbf{n}\cdot\nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}_\mathrm{v}\right) = \mathbf{n}\cdot\mathbf{Q}_\mathrm{v} & \text{(on S)} \ . \\ \mathbf{n}\cdot\mathbf{Q}_\mathrm{v} = 0 & \text{(on exterior wall } W_\mathrm{v}\text{)} \end{cases} \tag{3}$$

To proceed, a flux coordinate system (α,ψ,θ) with $\alpha = \zeta - q\theta$ the field line label, ψ the poloidal flux and θ the magnetic coordinate, is introduced and the plasma perturbation is decomposed in three perpendicular components:

$$\boldsymbol{\xi} = X\frac{\nabla\psi}{|\nabla\psi|^2} + U\frac{\nabla\psi\times\mathbf{B}}{B^2} + W\mathbf{B} \ . \tag{4}$$

and similarly for the vacuum magnetic perturbation. The perturbations are subsequently decomposed in Fourier Modes in the coordinates (α,θ) with mode numbers n and m, both $\gg 1$. However, since high n modes are considered, these modes are required to follow the field lines to first order, which is reflected in the fact that the modes only couple in the magnetic coordinate θ and not in the field line label α. This is the 3D high n equivalent of the fact that in 2D the toroidal coordinate is negligible. Note that θ is *not* the poloidal coordinate, but the coordinate along the magnetic field, which means that the equilibrium is *not* reduced to a toroidal cross section. In other words, the equilibrium information is preserved by the ability to move along the magnetic field lines (which are expected to cover the complete flux surface).

By then Euler minimizing first the parallel component Z, sound waves are eliminated, which are represented by the term proportional to γ. Secondly, the geodesic component U is Euler minimized by avoiding magnetic field line bending. After some calculation, this resuls in an expression relating U_m to X_m in the form of a linear operator:

$$U_m(X_m) = \left(-\Theta^\zeta + \frac{m}{n}\Theta^\theta + \frac{\mathrm{i}}{n}\frac{\partial}{\partial\psi}\right)X_m + \frac{\mathrm{i}}{n}Q_m(X_m) \,, \tag{5}$$

with Θ^i the ratio of two contravariant metric factors: $\Theta^i = \frac{g^{\psi,i}}{g^{\psi,\psi}} = \frac{\nabla\psi\cdot\nabla u^i}{\nabla\psi\cdot\nabla\psi}$, i the imaginary unit and Q_m respresenting the contribution of order $O\left(n^{-1}\right)$, given by:

$$Q_m(X_m) = \left[\frac{B_\alpha q' + \mathcal{J}\mu_0 p'}{B_\theta} + \left(-\Theta^\zeta + \Theta^\theta\frac{m}{n}\right)Q^1_m + \frac{nq-m}{n}\frac{\mathcal{J}\mathbf{B}\cdot\nabla\psi\times\nabla\Theta^\theta}{B_\theta} + \frac{nq-m}{n}\frac{B_\alpha}{B_\theta}\frac{\partial}{\partial\psi}\right]X_m \,, \tag{6}$$

with $\mathcal{J}$ the Jacobian and B_i the covariant components of the magnetic field.

The minimized geodesic component can now be inserted back into the expression for the plasma potential energy, as well as the kinetic energy. Secondly, for high n modes, it can be shown that the minimal potential energy due to the plasma-vacuum interface is zero, to avoid large stabilization. The vacuum potential energy, lastly, can be reduced to a term that depends only on the normal component of the plasma perturbation at the interface ξ_n, as shown also in equation 1.

The result of the minimization, $\delta W_\mathrm{p}[\boldsymbol{\xi}] + \delta W_\mathrm{v}[\mathbf{Q}_\mathrm{v}] = \Lambda\delta I[\boldsymbol{\xi}]$, then yields a compact, Hermitian formulation:

$$\boxed{(\mathbf{X}^*)^T\ (\mathbf{P} - \Lambda\mathbf{K})\ \mathbf{X} = 0 \,,} \tag{7}$$

where $\mathbf{X} = \left(X_m e^{-\mathrm{i}m\theta}\right)^T$ contains the M modes in m and the components of the tensor P are given by:

$$P_{k,m} = \widetilde{PV}^0_{k,m} - \frac{\mathrm{i}}{n}\frac{\overleftarrow{\partial}}{\partial\psi}\widetilde{PV}^{1*}_{m,k} + \widetilde{PV}^1_{k,m}\frac{\mathrm{i}}{n}\frac{\overrightarrow{\partial}}{\partial\psi} - \frac{\mathrm{i}}{n}\frac{\overleftarrow{\partial}}{\partial\psi}\widetilde{PV}^2_{k,m}\frac{\mathrm{i}}{n}\frac{\overrightarrow{\partial}}{\partial\psi} \tag{8}$$

and similarly for $\mathbf{K}$. The coefficients $\widetilde{PV}^i_{k,m}$ are Hermitian and listed in [7] and the direction of the arrows indicate whether the derivatives work on the left or on the right.

By means of Euler minimization with respect to the modes X^*_k of the vector $(\mathbf{X}^*)$, a system of M ordinary, second order differential equations is obtained. The boundary terms resulting from this minimization, as well as the terms representing the vacuum potential energy, yield boundary conditions at the plasma-vacuum interface. The problem is then completely described by considering external modes that vanish in the interior of the plasma.

This system of equations has to be solved, yielding the eigenvalue Λ and the eigenvectors that describe the stability of the system. Alternatively, by introducing the coordinate transformation $\theta_{3\mathrm{D}} = \frac{1}{q}\int^{\chi_{2\mathrm{D}}}\frac{\mathcal{J}_{2\mathrm{D}}B_{\zeta_{2\mathrm{D}}}}{R^2}$, the results from [8] can be obtained directly, where the subscript $_{2\mathrm{D}}$ indicates an axisymmetric quantity. This represents an alternative derivation that is much easier than the original one [1]. The numerical code ELITE is based directly on this theory.

The PB3D code: numerical aspects

A numerical code, called PB3D (Peeling - Ballooning in 3D) is currently under development. This code will solve the system of equations described in the previous section, making use of finite differences in the radial direction.

The code is written in FORTRAN, with performance in mind. One of the features supporting this is the usage of the VMEC system of coordinates, in which the coordinates are deformed in such a way that the size of a Fourier base in these coordinates, describing the equilibrium, is as small as possible. The downside is the fact that the VMEC coordinate system is generally not straight.

Furthermore, PB3D will be parallelized. This will enable the usage of the code in parameter studies so that the domains in which more accurate, but also slower codes should be used, can be determined more easily.

Conclusions

To fill a gap in the current understanding of 3D effects in magnetic configurations, which are important for many phenomena such as the suppression of ELMS and the limiting behavior of the pedestal in the H-mode, a new theory has been developed. This theory is based on linear ideal MHD and constitutes a coupled system of Hermitian, second order linear differential equations that have to be solved to determine the stability of a 3D plasma-vacuum system.

A new code, PB3D, is under development, which aims to accurately and quickly solve this system of equations so that it can be used for parameter studies, indicating the revant parameter regions where more accurate, but slower codes are necessary.

This will improve the understanding of 3D effects which are important for the next generations of magnetic devices.

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