Nonlinear growth of tearing modes: validating the generalized Rutherford equation

J. Heres, J. Pratt and E. Westerhof

FOM Institute DIFFER, Dutch Institute for Fundamental Energy Research, 3430 BE Nieuwegein, The Netherlands, www.differ.nl

Introduction

The nonlinear growth of neoclassical tearing modes (NTMs) in tokamaks is commonly discussed in the framework of the generalized Rutherford equation (GRE) [1, 2]. The GRE is obtained by averaging the current diffusion equation for the dominant Fourier harmonic of the perturbation over the island region and matching to the linear, ideal MHD solution outside the magnetic island. This way the GRE discards all information on the detailed structure of the island: possible asymmetries and contributions from higher Fourier harmonics to the perturbation. Results from the GRE qualitatively match experimental observations of NTM growth and their stabilization by electron cyclotron current drive (ECCD) [3]. To anticipate requirements for the suppression of NTMs by ECCD in future machines such as ITER, the GRE is being used extensively. However, the neglect of the detailed interior structure of the island, in particular, the limitation to a single dominant Fourier harmonic of the perturbation might break down in the case of localized ECCD, where the higher Fourier harmonics of the EC-driven current can become significant [4].

We have developed a 2D reduced MHD code with the specific aim of validating the GRE and its underlying assumptions and studying the influence of higher Fourier harmonics in the case of ECCD. In this contribution we present the first results obtained with this new code.

A 2D reduced MHD code

By focusing on a layer directly around a single magnetic island chain in a tokamak localized at the resonant radius r_s, the magnetic island evolution can be described by the set of 2D reduced MHD equations for the helical magnetic flux ψ and the potential φ:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\psi = \eta j - E, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\nabla^2\varphi = \mathbf{B}\cdot\nabla j + \nu\nabla^4\varphi, \tag{2}$$

where η is the resistivity, ν the viscosity, the current density $j \equiv \nabla^2\psi$, the magnetic field $\mathbf{B} = \hat{e}_z \times \nabla\psi + B_z\hat{e}_z$, and the velocity $\mathbf{v} = \hat{e}_z \times \nabla\varphi$. In line with assumptions made in the derivation of the Rutherford equation, the equilibrium flux function is approximated by the leading term of its Taylor expansion around the resonant surface, i.e. $\psi_{\mathrm{eq}}(x) = \frac{1}{2}x^2\psi''_{\mathrm{eq}}$, where $x \equiv r - r_s$ and the prime indicates the derivative with respect to x. Note that in normal tokamak equilibria with a positive toroidal current, the equilibrium shear ψ''_{eq} is negative. The constant E in the equation for the flux evolution keeps the equilibrium flux constant in time. The equilibrium potential is set to zero. The perturbations to both flux and potential are written in terms of Fourier series

$$\tilde{\psi}(x,\xi) = \sum_{k=-\infty}^{+\infty} \psi_k(x)e^{\mathrm{i}k\xi} \qquad \text{and} \qquad \tilde{\varphi}(x,\xi) = \sum_{k=-\infty}^{+\infty} \varphi_k(x)e^{\mathrm{i}k\xi}. \tag{3}$$

Note, that due to the symmetry of the equations in ξ, we can specify $\psi_{-k}(x) = \psi_{+k}(x)$ and $\varphi_{-k}(x) = -\varphi_{+k}(x)$. The radial boundary conditions for the flux are given in terms of the step in the logarithmic derivative over the simulated radial domain $[-L : +L]$ of each Fourier component of the perturbation. This way the boundary conditions are specified in terms of the tearing stability parameters Δ'_k of each Fourier harmonic:

$$\frac{\psi'_k(\pm L)}{\psi_k(\pm L)} = \pm\frac{\Delta'_k}{2}. \tag{4}$$

The corresponding boundary condition for the potential is obtained from linear ideal MHD as

$$\frac{\varphi'_k(\pm L)}{\varphi_k(\pm L)} = \pm\frac{\Delta'_k}{2} \mp \frac{1}{L}. \tag{5}$$

The stability of a mode is determined in the first place by the equilibrium profiles outside the narrow layer that is simulated. A positive value of Δ'_k means that this particular Fourier harmonic is unstable. In that case, linear theory predicts an exponential growth rate γ of the flux perturbation which is given by [5]

$$\gamma = 0.55(\Delta')^{4/5}\eta^{3/5}(k\psi''_{\mathrm{eq}})^{2/5}. \tag{6}$$

In practical cases only the dominant Fourier harmonic $k = \pm 1$ will be unstable while all higher harmonics are typically stable.

The code that we have developed uses finite differences in the radial direction x and Fourier decomposition in the helical angle $\xi \equiv m\theta - n\phi$ where m and n are the poloidal and toroidal mode numbers, respectively. The time stepping is performed using a fourth order Runge-Kutta scheme. Only the dominant Fourier harmonic is assumed to be unstable and the corresponding positive value of Δ'_1 is specified at input. All harmonics $|k| \leq 2$ are taken to be stable with $\Delta'_k = -\frac{2|k|m}{r_s}$.

The (generalized) Rutherford equation, is obtained by assuming that (i) the mode is fully determined by the leading harmonic $\tilde{\psi} = 2\psi_1 \cos\xi$, ($ii$) ψ_1 is approximately constant in x, (*iii*) the island is very narrow such that the perturbed current density is $\tilde{j} = \partial^2\tilde{\psi}/\partial x^2$, and (*iv*) inertia can be neglected such that $\mathbf{B}\cdot\nabla j = 0$ which implies that the current density becomes a flux function. The fourth assumption means that we can take the flux surface average of Ohm's law (1) $\langle \partial\tilde{\psi}/\partial t\rangle = \eta j$. When we integrate Ampére's law multiplied with $\cos\xi$ over the entire domain covering the magnetic island and substitute from Ohm's law for ηj we obtain

$$\frac{1}{2}\eta\Delta'\psi_1 = \eta\int_{-\infty}^{+\infty}\mathrm{d}x\oint\mathrm{d}\xi\,\frac{\partial^2\tilde{\psi}}{\partial x^2}\cos(\xi) = \int_{-\infty}^{+\infty}\mathrm{d}x\oint\mathrm{d}\xi\left\langle\frac{\partial\tilde{\psi}}{\partial t}\right\rangle\cos(\xi) = g_1 w\frac{\partial\psi_1}{\partial t}. \quad (7)$$

The left hand side is obtained by matching to the boundary conditions. The right hand side contains the island size w and a geometrical constant g_1, which has the value 0.82 [6]. Using the equation for the island width, $w = 4\sqrt{2\psi_1/|\psi''_{\mathrm{eq}}|}$, we obtain the generalized Rutherford equation:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\eta}{g_1}\left(\Delta' + \Delta'(J_{\mathrm{bs}}) + \Delta'(J_{\mathrm{ECCD}}) + \ldots\right), \quad (8)$$

where the additional terms on the right originate from noninductive current contributions that alter Ohm's law [2], like perturbations of the bootstrap current J_{bs} or electron cyclotron driven currents J_{ECCD}.

First results

In order to verify the code we check that the growth rate of the island in the linear regime is exponential and satisfies the theoretical expression (6). Typical input parameters are equilibrium shear $\psi''_{\mathrm{eq}} = -5\times 10^5$ s^{-1}, stability parameter $\Delta' = 1.00$ ... 100. m^{-1}, and resistivity $\eta = 0.01$ ... 1.00 m^2/s. All results presented have been obtained using a limited range of Fourier modes of $-3 \le k \le +3$. Figure 1 shows the exponential growth rate γ for (left) a scan in resistivity at $\Delta' = 1.0$ m^{-1} and (right) a scan in Δ' at $\eta = 0.01$ m^2/s. The growth rate calculated by our 2D reduced MHD code agrees well with the theoretical scaling of equation (6). The deviation at large values of Δ' might be explained by a break down of the constant-ψ approximation.

In a second test we continued the simulations well into the nonlinear regime. For a simulation with $\Delta' = 1.0$ m^{-1} and $\eta = 0.01$ m^2/s, we present the results in Figure 2. The left frame shows the evolution of the island size as a function of time. After an initial exponential growth the growth rate is seen to slow down from about $t = 1.5$ s, and then continues to grow proportional to time as expected from the GRE. In order to compare quantitatively to the predictions of the GRE the right frame shows the evolution of $(dw/dt)/(\eta\Delta')$, which according to the GRE is expected to attain a constant value of $(1/g_1) = 1.22$. Indeed the code results reach a constant value of dw/dt very close to the predictions of the GRE. Similar results are obtained over a wide range of parameters with the larger values of Δ' showing a faster than predicted growth also in the nonlinear regime.

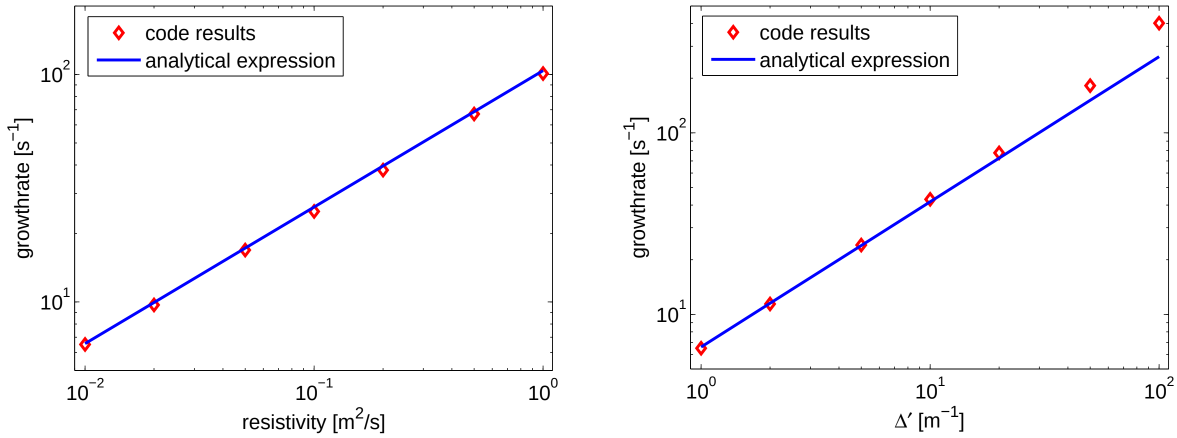

Figure 1: Scaling of the exponential growth rate γ of the dominant harmonic of the helical flux perturbation with (left) resistivity and (right) stability index Δ'. The code results are represented by red diamonds and the blue curves represent the theoretical expression (6).

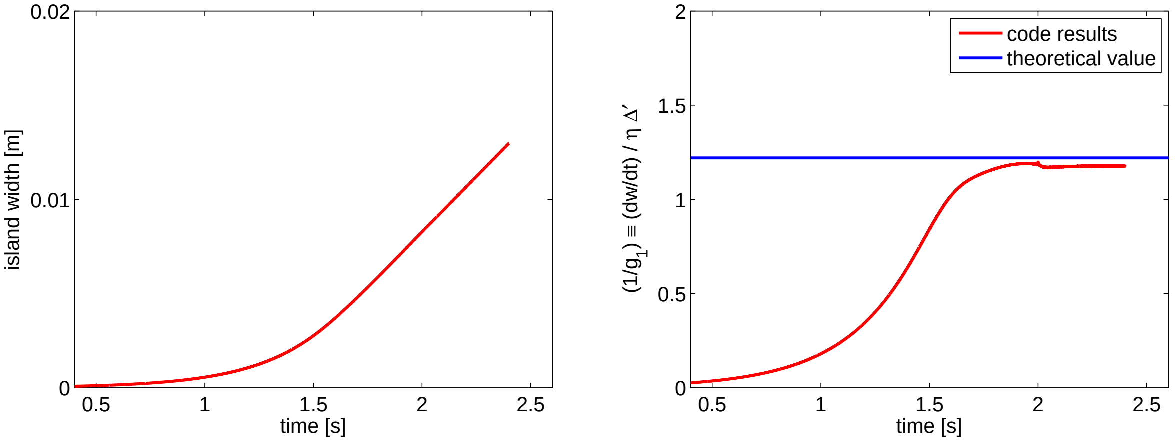

Figure 2: Results of a simulation with $\Delta' = 1.0\ \mathrm{m}^{-1}$ and $\eta = 0.01\ \mathrm{m}^2/\mathrm{s}$. The left frame shows the island size w as a function of time. The right frame compares the simulated rate of change of the island size (red) to the predictions of the GRE (blue).

Acknowledgment. This work has been performed in the framework of the NWO-RFBR Centre of Excellence (grant 047.018.002) on Fusion Physics and Technology. This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement number 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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