An iterative method for including spatial dispersive effects in global wave solvers using FEM decomposition

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Modelling of the wave field for RF-heating is a challenging task because of the spatial dispersive nature of magnetised plasmas and the co-existence of different waves. The issue can be handled by Fourier decomposing the wave equation [1]. Hybrid methods using Fourier decomposition in the toroidal and the poloidal directions and FEM (finite element methods) decomposition across the flux surfaces are commonly used. The Fourier decomposition leads to dense matrixes that become time consuming to invert. FEM methods have the advantages of producing local decomposition which, in general, is faster to invert, but for which it is more difficult to include spatial dispersive effects. Lately methods to include or correct for spatial dispersive effects have been developed [2-4]. Here a method is proposed using operator splitting to correct for spatial dispersive effects where the correction term appears as a source function in the wave equation representing induced current by the wave field. The wave equation is then solved by means of iteration with the source function calculated from the previous wave field by separating it locally into planar waves and using susceptibility tensors developed for hot uniform plasmas. The method enables modelling of upshift, finite Larmor radius effects to all order and modelling of kinetic waves with standard global wave solver without expanding the susceptibility tensors into higher order differential operators or solving integral equations.

Waves launched into a dispersive medium by an antenna can be described by

$$\nabla\times\nabla\times\mathbf{E}-\left(\frac{\omega}{c}\right)^2\mathbf{E}=i\omega\mu_0\mathbf{J}_{ant}+i\omega\mu_0\mathbf{J}_{ind} \qquad (1)$$

$$\mathbf{J}_{ind}=L\left(\mathbf{E}\right), \qquad (2)$$

where $\mathbf{J}_{\mathrm{ant}}$ is the antenna current, $\mathbf{J}_{\mathrm{ind}}$ the current induced in the plasma by the electric wave field $\mathbf{E}$, here we assume $L\left(\mathbf{E}\right)$ to be a linear operator. The induced current density depends on interactions between the charged particles and the wave field before the actual time, which makes the calculations of the induced current non-trivial. In the Fourier space (ω, $\mathbf{k}$) the current induced by a fluctuating electric field can straightforwardly be defined by

$$J_{ind,j}\left(\omega,\mathbf{k}\right)=\sigma_{jl}\left(\omega,\mathbf{k}\right)E_l\,. \tag{3}$$

When transforming the response into the real space the local current is then given by

$$J_{ind,j}\left(t,\mathbf{x}\right)=\hat{\sigma}_{jl}\left(t,\mathbf{x}\right)*E_l\left(t,\mathbf{x}\right), \tag{4}$$

where * here denotes the convolution between the electric field and $\hat{\sigma}_{jl}\left(t,\mathbf{x}\right)$, the inverse Fourier transform of the $\sigma_{jl}\left(\omega,\mathbf{k}\right)$. The convolution describes non-localized wave-particle interactions. The dominating spatial dispersive effects in the ion cyclotron frequency range, ICRF, arise from the gyro motion and the spatial localisation of the resonances. The gyro motion affects the absorption and introduces new waves, the kinetic Alfvén waves and the Bernstein waves, with length scale of the order of the gyro radius. The spatial dispersive effects complicate the problem by that the response depends on the wave vector and the possibility of co-existence of several waves having different response tensors. Traditionally the problem is solved by substituting Eq. (2) into Eq. (1). Because of the complicated structure of the operator describing the induced current, Green et al [4] proposed to solve Eqs. (1, 2) through iterations, so called operator splitting. To take into account the Doppler broadening they calculated the induced current by integrating the change in velocity along unperturbed particle orbits in the calculated wave field. To calculate the wave field by means of iteration required special iteration methods. The minimum polynomial extrapolation method, MPE, provided converged results.

Calculation of the current by following the orbit in the wave field is time consuming. To avoid such calculations we assess if the non-local effects can be included in a more simplified way. Wave-particle interactions at high frequency in inhomogeneous plasmas such as in a tokamak differ conceptually from that in homogeneous plasmas. When the decorrelation time is shorter than the bounce time, the Doppler shifted resonance becomes the resonance and not the bounce resonances. Assuming that the wave field can be decomposed locally into different coherent planar waves a particle moving along a magnetic field line may then resonate with several planar waves. Because of the rapid variation of the phase difference between the wave and particle motion, interactions with a single wave can be calculated with the stationary phase method. For simplicity we assume the interactions to be well localised and to take place at the stationary phase points. A single particle interacts then only with one wave at the same location described by (t_k, z_k) provided $\upsilon_{\|} \neq 0$ and the parallel wave numbers $k_{\|,k}$ differ for the different waves. Between the interactions the particle phase may be decorrelated e.g. due to collisions. The change in the velocity at a point (t_0, z_0) of a particle having a velocity $\mathbf{v}_0$ depends then on all previous interactions, which is obtained by integrating the force caused by the wave-particle interaction along the particle orbit. Taking into account the decorrelation one obtains

$$\Delta\mathbf{v}\left(z_0,t_0,\mathbf{v}_0,\mathbf{E}\right)=\sum_k \exp i\phi_k\, Ze/m\left(\mathbf{E}+\mathbf{v}_0\times\mathbf{B}\right)_k \Delta t_k\delta_k\,, \tag{5}$$

where $\mathbf{B}$ is the magnetic induction caused by the wave field and $Ze/m\left(\mathbf{E}+\mathbf{v}_0\times\mathbf{B}\right)_k \Delta t_k$ is the effective acceleration at the resonance z_k at the time t_k, δ_k is the effective correlation of this change remaining at the time t_0 and ϕ_k is the change in phase between the resonance and the point (t_0, z_0). The interaction time, $2\Delta t$, can be defined as twice the time it takes for the phase difference to increase with $\pi/2$ and the corresponding distance, $2\Delta l$, which gives

$$2\Delta t=2\left(\frac{\pi qR}{rn\omega_c\upsilon_{||}\sin\theta}\right)^{1/2} \qquad 2\Delta l=2\upsilon_{||}\Delta t=2\left(\frac{\pi qR\upsilon_{||}}{rn\omega_c\sin\theta}\right)^{1/2}, \tag{6a, 6b}$$

where q is the safety factor, R major radius, r minor radius θ the poloidal angle. A considerable simplification is obtained, if we assume the decorrelation time to be longer than the resonant interaction time $2\Delta t$, but sufficiently short for the different interactions to be decorrelated. Then only the contribution from the wave resonating at z_0 remains, which gives $\delta_k = 0$ if $z_k \neq z_0$ and $\delta_k = 1$ if $z_k = z_0$.

For completely decorrelated interactions we need only to take into account the variations of the wave field along the magnetic field over a distance $2\Delta l$ to resolve the resonances i. e. to calculate the poloidal upshift, which can then be done by decomposing the wave field locally into planar waves. The response of a bi-Maxwellian plasma can then be approximated using the plasma dispersion function for each planar wave.

The effects of the gyro motion can be included by integrating the acceleration of the particle by the wave field over a gyro orbit. The induced current in a non-uniform plasma for monochromatic waves is given by

$$J_{ind,j}\left(\omega,\mathbf{k},\mathbf{x}\right)=\sigma_{jl}\left(\omega,\mathbf{k}\right)E_l-i\frac{E_l}{E_{0,l}}\frac{\partial}{\partial k_s}\sigma_{jl}\left(\omega,\mathbf{k}\right)\frac{\partial E_{0,l}}{\partial x_s}+\tilde{\sigma}_{jl}\left(\omega,\mathbf{k},E_l\right) \tag{7}$$

where $E_l=E_{0,l}\left(\mathbf{x}\right)\exp i\left(\int\mathbf{k}d\mathbf{x}-\omega t\right)$ and $\sigma_{jl}\left(\omega,\mathbf{k}\right)$ is the conductivity tensor in a homogeneous hot plasma and $\tilde{\sigma}_{jl}\left(\omega,\mathbf{k},E_l\right)$ is a higher order correction term ensuring that divergence of the energy flux vanishes in absence of absorption.

Because the important spatial dispersive effects in the ICRF are pseudo-localised the wave field needs only to be locally decomposed of the order of a gyro radius for the gyro motion and $2\Delta l$ for the Doppler shift. The advantage with a local decomposition is that we can regard the different interactions to be uncorrelated, which requires a quasi-linear normalisation of the wave field. An ordinary Fourier decomposition requires the modes to be correlated. The continuous Morlet wavelets are well suited for such a decomposition, for which the transform is defined by

$$S_{m,n} = \int_{-\infty}^{+\infty} s(x')\psi^{*}_{m,n}(x')\,dx' , \tag{8}$$

where * denotes the complex conjugate and $\psi_{\rm m,n}$ is related to the mother wavelet ψ by

$$\psi_{\rm m,n}(x) = a^{-m/2}\psi\left(\frac{x-nb}{a^m}\right), \tag{9}$$

where $\psi(\mathrm{x}) = \left(e^{-i\omega_0 x} - e^{-\frac{\omega_0^2}{2}}\right)e^{-\frac{x^2}{2}}$. The effective number of oscillations of the wavelet is defined by ω_0. The inverse transform is given by

$$s(\mathrm{x}) = k_\psi \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \psi_{m,n}(x) S_{m,n} . \tag{10}$$

When modelling the induced current the derivative $\partial E_{0,l}/\partial x_s$ in Eq. (7) is calculated by taking the derivative on the envelop function of the wavelet given by. Modelling the upshift with FEM codes for the fast magnetosonic wave requires that the wave field is decomposed into wavelets in the poloidal direction on each flux surface. As an initial approximation the current density obtained from a hot dielectric tensor with $k_{||} = n_\phi/R$ can be used with $\mathbf{k}_\perp$ perpendicular to the flux surfaces. Note the MPE can be initialized by using a few initial current distributions obtained e.g. from different $k_{||}$ values in the dielectric tensor. Modelling slow kinetic waves requires wavelet decomposition both in the poloidal direction and across the magnetic surface where the current distribution has to be resolved within a gyro orbit.

References

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