Finite-aspect-ratio effects on neoclassical transport coefficients, revisited

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The neoclassical moment method is improved by increasing the accuracy of approximation to the linearized Fokker-Planck collision operator. In this paper, we apply this improved method to the calculation of ion flow velocity for a plasma with one impurity species.

Let us write the perturbed distribution function as $f_{a1} = -(Iv_{\parallel}/\Omega_a) f'_{a0} + g_a$ in an axisymmetric magnetic field $B = I(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi$, where $v_{\parallel} = B \cdot v/B$, Ω_a is the Larmor frequency, $2\pi\psi$ is the poloidal flux and φ is the toroidal angle. Then, in the banana regime, the function g_a for trapped particles ($\lambda > \lambda_c$) is identically zero, where $\lambda = (1 - v_{\parallel}^2/v^2)/B$ and $\lambda_c = 1/B_{\max}$. The analytic function g_a for passing particles ($\lambda < \lambda_c$) can be obtained from the solubility condition by using some sort of approximation to the linearized Fokker-Planck collision operator $C_{ab}(f_{a1}, f_{b1})$.

We introduce the following approximate collision operator:

$$C_{ab}(f_{a1}, f_{b1}) \simeq \nu_D^{ab}(v)\mathscr{L}(f_{a1}) + \sum_{l=0}^{3} P_l(\xi)\hat{C}_{ab}^l(f_{a1}^1, f_{b1}^1) \tag{1}$$

with

$$\hat{C}_{ab}^l(f_{a1}^1, f_{b1}^1) = C_{ab}^l(f_{a1}^1, f_{b1}^1) + \frac{l(l+1)}{2}\,\nu_D^{ab}(v)\,f_{a1}^l, \tag{2}$$

where $f_{a1}^l = (l+1/2)\int_{-1}^{1} P_l(\xi)\,f_{a1}d\xi$, $C_{ab}(P_l(\xi)\,f_{a1}^l(v), P_l(\xi)\,f_{b1}^l(v)) = P_l(\xi)C_{ab}^l(f_{a1}^l(v), f_{b1}^l(v))$, $\xi = v_{\parallel}/v$, $\mathscr{L}$ is the pitch-angle scattering operator, the deflection collision frequency $\nu_D^{ab}(v) = \nu_{ab}[\mathrm{erf}(v/v_b) - G(v/v_b)](v_a/v)^3$ with $G(x) = [\mathrm{erf}(x) - (2x/\sqrt{\pi})\exp(-x^2)]/(2x^2)$ and $\nu_{ab} = 4\pi n_b e_a^2 e_b^2 \log\Lambda\,/(m_a^2 v_a^3)$, thermal velocity $v_a = \sqrt{2T_a/m_a}$, and n_a and T_a are the number density and temperature, and m_a and e_a are the mass and charge. Using this approximate collision operator, we can obtain the distribution function for passing particles in the form

$$g_a = \frac{1}{2}\frac{\sigma}{\nu_D^a(v)}\int_{\lambda}^{\lambda_c}\frac{d\lambda}{\langle\sqrt{1-\lambda B}\rangle}G_{a1}(v) + \frac{5}{8}\frac{\langle B^4\rangle}{\langle B^3\rangle}\frac{\sigma}{\nu_D^a(v)}\int_{\lambda}^{\lambda_c}\frac{\lambda d\lambda}{\langle\sqrt{1-\lambda B}\rangle}G_{a2}(v), \tag{3}$$

where $\sigma = v_{\parallel}/|v_{\parallel}|$, $\nu_D^a(v) = \sum_b \nu_D^{ab}(v)$, $\langle\cdot\rangle$ denotes the flux-surface average,

$$G_{a1} = \sum_b\left[\hat{C}_{ab}^1(K_{a1}, K_{b1}) + \hat{C}_{ab}^3(K_{a3}, K_{b3}) - C_{ab}^1\left(\frac{IB}{\Omega_a}v f'_{a0}, \frac{IB}{\Omega_b}v f'_{b0}\right)\right] \tag{4}$$

and

$$G_{a2} = \sum_b\left[\frac{7}{3}F_t\,\hat{C}_{ab}^3(K_{a1}, K_{b1}) - \hat{C}_{ab}^3(K_{a3}, K_{b3})\right] \tag{5}$$

with $F_t = 1 - \langle B^3 \rangle^2 / \langle B^2 \rangle \langle B^4 \rangle$. The function $K_{a1}(v) \equiv \langle B g_a^1 \rangle$ is determined by the equation

$$\frac{f_t}{f_c} v_D^a(v) K_{a1} - \sum_b C_{ab}^1(K_{a1}, K_{b1}) - \Delta_t \sum_b \hat{C}_{ab}^3 (K_{a1}, K_{b1}) = - \sum_b C_{ab}^1 \left(\frac{IB}{\Omega_a} v f_{a0}', \frac{IB}{\Omega_b} v f_{b0}' \right) \quad (6)$$

with

$$\Delta_t = \frac{7}{3} \left(F_t \tilde{f}_c + \tilde{f}_t \bar{f}_t \right), \quad (7)$$

where $f_t = 1 - f_c$, $\tilde{f}_t = 1 - \tilde{f}_c$, $\bar{f}_t = \tilde{f}_t + F_t \tilde{f}_c$,

$$f_c = \frac{3}{4} \langle B^2 \rangle \int_0^{\lambda_c} \frac{\lambda d\lambda}{\langle \sqrt{1 - \lambda B} \rangle}, \quad (8)$$

$$\tilde{f}_c = \frac{15}{16} \frac{1}{f_c} \frac{\langle B^4 \rangle \langle B^2 \rangle}{\langle B^3 \rangle} \int_0^{\lambda_c} \frac{\lambda^2 d\lambda}{\langle \sqrt{1 - \lambda B} \rangle}, \quad (9)$$

and the function $K_{a3}(v) \equiv \langle B g_a^3 \rangle$ is approximately obtained from the relation

$$K_{a3}(v) \simeq \frac{7}{3} \bar{f}_t K_{a1}(v). \quad (10)$$

The flow velocity is expressed in the form

$$u_a = u_{a\theta} B - \frac{T_a}{m_a \Omega_a} \left(\frac{p_a'}{p_a} + \frac{e_a \Phi'}{T_a} \right) R^2 \nabla \varphi, \quad (11)$$

where the poloidal flow $u_{a\theta}$ is written in terms of the function $K_{a1}(v)$ as $u_{a\theta} = (4\pi/3n_a) \times (1/\langle B^2 \rangle) \int_0^\infty dv v^3 K_{a1}$.

We next explicitly calculate the poloidal flows of primary ions and impurities. Let us expand the function $K_{a1}(v)$ in a series of the associate Laguerre polynomials of order 3/2 and retain only the first and second terms: $K_{a1}(v) \simeq (m_a/T_a) \langle B^2 \rangle v [u_{a\theta} - (2/5) q_{a\theta}/p_a \, (5/2 - v^2/v_a^2)] f_{a0}$, where $p_a = n_a T_a$ and f_{a0} is the Maxwell distribution function. Inserting this expansion for K_{a1} into Eq.(6) and taking velocity moment with respect to v^3 and $v^3(5/2 - v^2/v_a^2)$ lead to a set of coupled algebraic equations for $u_{a\theta}$ and $q_{a\theta}$. From here, the subscripts i and I represent the primary and impurity ions. Assuming that the primary ions are in the banana regime and using the smallness of the mass ratio m_i/m_I, we solve this coupled equations to find the poloidal flows $u_{i\theta}$ and $u_{I\theta}$:

$$\begin{bmatrix} u_{i\theta} \\ u_{I\theta} \end{bmatrix} = \frac{IcT_i}{e_i \langle B^2 \rangle} \sum_{k=1}^{4} \begin{bmatrix} u_{ik} \\ u_{Ik} \end{bmatrix} A_k, \quad (12)$$

where

$$\begin{bmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{i4} \end{bmatrix} = \frac{1}{D_i} \begin{bmatrix} \beta_1 s_1 \\ \tilde{\mu}_{i2} s_2 - (3/2) \tilde{\mu}_{i3} s_1 \\ \beta_1 s_3 \\ -5 \alpha \mu_{I2} \delta \delta_T \beta_1 / D_I \end{bmatrix}, \quad (13)$$

$$\begin{bmatrix} u_{I1} \\ u_{I2} \\ u_{I3} \\ u_{I4} \end{bmatrix} = \frac{\delta}{D_i D_I}\left(\mu_{I3}+\sqrt{2}+\frac{15}{2}\delta\right)\begin{bmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} - \frac{\mu_{I2}}{D_I}\begin{bmatrix} 0 \\ 0 \\ \sqrt{2}+15\delta/2 \\ 5\delta\delta_T \end{bmatrix}, \quad (14)$$

and the thermal forces are defined by $A_1 = p_i'/p_i - (e_i/e_I)(T_I/T_i)(p_I'/p_I)$, $A_2 = T_i'/T_i$, $A_3 = (e_i/e_I)(T_I'/T_i)$, $A_4 = (T_I/T_i)(e_i/e_I)(p_I'/p_I + e_I\Phi'/T_I)$. The parameters in (13) and (14) are defined as $\alpha = n_I e_I^2/(n_i e_i^2)$, $\delta = (1/\alpha)\sqrt{m_i/m_I}/(T_I/T_i)^{3/2}$, $\delta_T = 1 - T_i/T_I$, $\beta_1 = \tilde{\mu}_{i3} - s_2 + (3/2)(\tilde{\mu}_{i2} - 3s_1/2)$, $\beta_2 = (9s_1/4 + s_2)\tilde{\mu}_{i1} + \tilde{\mu}_{i2}^2 - \tilde{\mu}_{i1}\tilde{\mu}_{i3}$, $\beta_3 = (3/2)(\tilde{\mu}_{i1}\tilde{\mu}_{i3} - \tilde{\mu}_{i2}^2) + (9s_1/4 + s_2)\tilde{\mu}_{i2}$, $\beta_4 = 9\tilde{\mu}_{i1}/4 + 3\tilde{\mu}_{i2} + \tilde{\mu}_{i3} - 9s_1/4 - s_2$, $\beta_5 = -5(\alpha/D_I)\mu_{I2}\delta\delta_T\beta_4$,

$$D_i = (\tilde{\mu}_{i1}+s_1)(\tilde{\mu}_{i3}-s_2) - \left(\tilde{\mu}_{i2} - \frac{3}{2}s_1\right)^2, \quad (15)$$

$$D_I = (\mu_{I1}+\delta)\left(\mu_{I3}+\sqrt{2}+\frac{15}{2}\delta\right) - \mu_{I2}\left(\mu_{I2}+5\delta_T\delta\right), \quad (16)$$

$$s_1 = \frac{\alpha}{D_I}\left\{\mu_{I1}\left(\mu_{I3}+\sqrt{2}+\frac{15}{2}\delta\right) - \mu_{I2}\left(\mu_{I2}+5\delta_T\delta\right)\right\}, \quad (17)$$

$$s_2 = \frac{1}{D_I}\left\{-(\mu_{I3}+\sqrt{2}+\frac{15}{2}\delta)\left[\sqrt{2}(\mu_{I1}+\delta)+\frac{13}{4}\alpha\mu_{I1}+\alpha\delta\right] + \left(\sqrt{2}+\frac{13}{4}\alpha\right)\mu_{I2}\left(\mu_{I2}+5\delta_T\delta\right)\right\}, \quad (18)$$

$$s_3 = -\frac{\alpha}{D_I}\mu_{I2}\left(\sqrt{2}+\frac{15}{2}\delta\right). \quad (19)$$

The viscosity coefficients for primary ions are written as

$$\begin{bmatrix} \tilde{\mu}_{i1} \\ \tilde{\mu}_{i2} \\ \tilde{\mu}_{i3} \end{bmatrix} = \begin{bmatrix} \mu_{i1} \\ \mu_{i2} \\ \mu_{i3} \end{bmatrix} - \Delta_t \begin{bmatrix} (\hat{C}_{ii}^3)_{00} \\ (-\hat{C}_{ii}^3)_{01} \\ (\hat{C}_{ii}^3)_{11} \end{bmatrix}, \quad (20)$$

where the conventional viscosity coefficients are given by $\mu_{i1} = (f_t/f_c)[\sqrt{2}+\alpha-\log(1+\sqrt{2})]$, $\mu_{i2} = (f_t/f_c)[-2\sqrt{2}-3\alpha/2+(5/2)\log(1+\sqrt{2})]$ and $\mu_{i3} = (f_t/f_c)[(39/8)\sqrt{2}+13\alpha/4 -(25/4)\log(1+\sqrt{2})]$, and the matrix elements of $\hat{C}_{ii}^3$ are calculated as follows: $(\hat{C}_{ii}^3)_{00} = -(1087/63)\sqrt{2} + (589/21)\log(1+\sqrt{2})$, $(\hat{C}_{ii}^3)_{01} = -(143/126)\sqrt{2} + (55/21)\log(1+\sqrt{2})$, $(\hat{C}_{ii}^3)_{11} = (50923/504)\sqrt{2} - (13625/84)\log(1+\sqrt{2})$. The viscosity coefficients for impurities in the plateau to Pfirsch-Schlüter regime are given by

$$\begin{bmatrix} \mu_{I1} \\ \mu_{I2} \\ \mu_{I3} \end{bmatrix} = \frac{\tau_{II}}{n_I}\frac{8\pi}{3}\int_0^\infty dv\, \frac{v^4}{v_I^2}\, v_D^I(v)\, \frac{f_{tI}^\star}{1+f_{tI}^\star/\hat{f}_{tI}}\, f_{I0}\begin{bmatrix} 1 \\ \frac{v^2}{v_I^2}-\frac{5}{2} \\ \left(\frac{v^2}{v_I^2}-\frac{5}{2}\right)^2 \end{bmatrix}, \quad (21)$$

where $\tau_{II} = 3\sqrt{\pi}/(4\nu_{II})$,

$$\hat{f}_{tI} = \frac{3}{5}\frac{\langle (b\cdot\nabla B)^2\rangle}{\langle B^2\rangle}\frac{v^2}{\nu_D^I(v)\,\nu_T^I(v)}, \qquad f_{tI}^\star = \frac{3\pi}{16}\varepsilon^2\frac{v}{Rq}\frac{1}{\nu_D^I(v)}, \tag{22}$$

$\nu_T^I(v) = \nu_{II}\left\{[\mathrm{erf}(v/v_I) - 3G(v/v_I)](v_I/v)^3 + 8(v_I/v)G(v/v_I) + (8/3\sqrt{\pi})\delta\right\}$, $\nu_D^I(v) = \nu_{II} \times \left\{[\mathrm{erf}(v/v_I) - G(v/v_I)](v_I/v)^3 + (4/3\sqrt{\pi})(T_i/T_I)\delta(v_I/v)^2\right\}$ and $f_{tI}^\star$ is obtained for a model magnetic field with circular flux surfaces, i.e., $B = B_0/(1+\varepsilon\cos\theta)$. The impurity viscosity coefficients in the banana regime are obtained in the form:

$$\begin{bmatrix} \mu_{I1} \\ \mu_{I2} \\ \mu_{I3} \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_{i1}(\alpha=0) \\ \tilde{\mu}_{i2}(\alpha=0) \\ \tilde{\mu}_{i1}(\alpha=0) \end{bmatrix} + \frac{f_t}{f_c}\frac{T_i}{T_I}\delta \begin{bmatrix} 2/3 \\ -2/3 \\ 5/3 \end{bmatrix}. \tag{23}$$

Finally we show the normalized poloidal flows of primary ions due to the thermal force A_2 in the model magnetic field with circular flux surfaces. The normalized flows u_{i2} for α = 0, 1 and 4 are plotted as a function of the inverse aspect ratio ε in Fig. 1. The impurity ions are assumed to be (a) in the plateau to Pfirsch-Schlüter regime and (b) in the banana regime. The normalized flows obtained by the conventional moment method are lager than those by our method by up to about 20 % in the range of intermediate aspect ratio. We also plot those conventional flows by dotted curves in Fig.1.

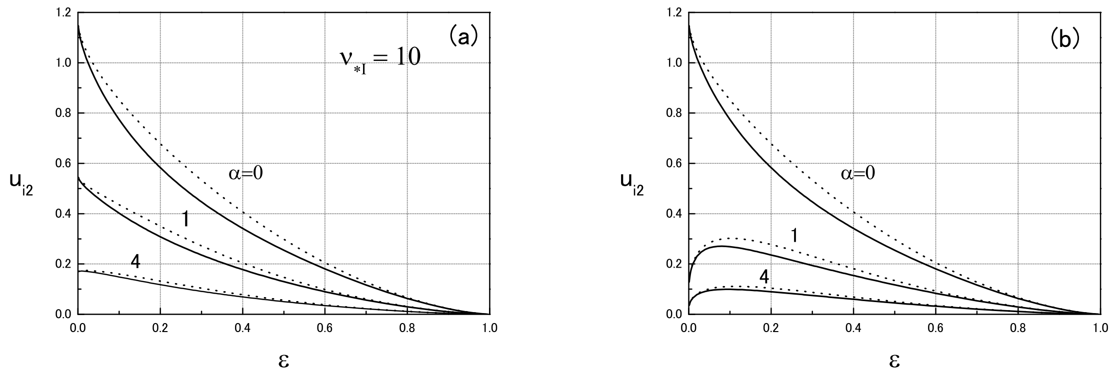

Figure 1: Normalized poloidal flow u_{2i} versus inverse aspect ratio ε. For comparison, the flows obtained by the conventional moment method are also plotted by the dotted curves. The parameter $\nu_{\star I}$ in (a) is defined by $\nu_{\star I} = (16/3\pi)(f_t/f_c)(1-\varepsilon^2)Rq/(\varepsilon^2 v_I \tau_{II})$.

References

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