# Equilibrium and stability with m=0 magnetic islands

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Axially symmetric $m = 0$ magnetic islands are studied in a cylindrical plasma with longitudinal magnetic field reversal. Applying 2D deformation on the equilibrium plasma boundary results in the breaking of magnetic surfaces topology into chain of islands. Axially symmetric equilibrium with islands are described by the Grad-Shafranov equation for the longitudinal flux function with a periodicity condition in $z$.

It was shown in [1] that axially symmetric equilibria with toroidal field reversal satisfying the Ohm's law do not exist and some of symmetry-breaking (e.g. helical) is required. The ohmic states are observed in 3D closed configurations as the self-organized helical reversed field pinch (RFP) equilibria [2]. So the $m = 0$ islands are 3D objects in realistic toroidal configurations. To model them in 2D we use force-free equilibria with a constant ratio between the current density and the magnetic field, $\mathbf{j} = \lambda \mathbf{B}$, across the plasma. In a cylinder it coincides with the Bessel function model (BFM) with finite current density at the plasma boundary [3].

Plasma stability with a chain of $m = 0$ magnetic islands is investigated for high aspect ratio RFP-relevant configurations. The unstructured grid MHD_NX code [4] is used to compute ideal MHD stability of 2D axially symmetric equilibria with arbitrary topology of magnetic surfaces.

**1 2D equilibrium and stability model** The starting point in 2D modeling is force-free axisymmetric equilibrium with the current density $\mathbf{j} = \lambda \mathbf{B}$ and prescribed plasma boundary oscillating around the 1D cylinder with minor radius $a$, $r_b(z) = a - d\ cos(2\pi z/z_p)$, and determined by the parameters $d, z_p$ controlling the size of the resulting magnetic island in $r$ and $z$ directions in case of field reversal. The choice of $z$-region consisting of $p$ periods, $0 \le z \le pz_p$, produces the chain of $p$ islands. The values $a = 1, d = 0.05, z_p = 2$ are mostly used below.

The 2D equilibrium magnetic field $\mathbf{B} = \nabla\psi \times \nabla\theta + f\nabla\theta$ features the $m = 0$ island(s) in place of field reversal for the cylindrical BFM magnetic field, $B_z(r) = J_0(\lambda r), B_\theta(r) = J_1(\lambda r)$ that happens when $\lambda r > 2.4$. In the 1D case the flux function $\psi$ is also known, $\psi_{cyl}(r) = rJ_1(\lambda r)/\lambda$. The boundary value of the cylindrical $\psi$ is used in 2D modelling as the boundary condition $\psi(r,z) = \psi_{cyl}(a), r = r_b(z)$ for numerical solution of the Grad-Shafranov equation

$$-r^2\nabla \cdot \left(\frac{\nabla\psi}{r^2}\right) = r^2 p' + ff' \qquad (1)$$

with $p' \equiv dp/d\psi = 0, f = \lambda\psi$ in the chosen force-free case. An increase of the value of $\lambda$ models a range of equilibria from "shallow" ($2.4 < \lambda a \le 2.7$, see Fig.1) to "deep" field reversal shifting respectively the magnetic islands deeper into the plasma. The corresponding cylindrical safety factor $q = (rB_z)/(RB_\theta) = (2\pi r d\psi/dr)/(fpz_p)$ edge/axis ratio would change from

$q_a/q_0 = -0.14$ when $\lambda a = 2.5$ to $q_a/q_0 = -2.26$ when $\lambda a = 3.3$. The axis value is given by $q_0 = 4\pi a/(\lambda p z_p)$.

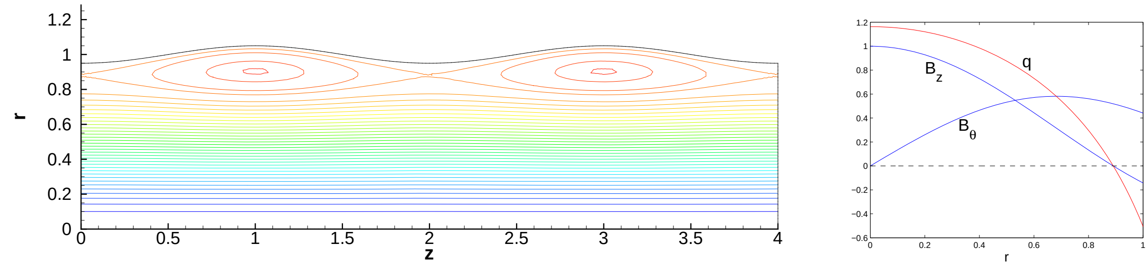


*Figure 1. The magnetic surfaces with two $m = 0$ islands (left), color corresponds to the levels of flux function $\psi$. "Shallow" field reversal, $\lambda = 2.7$. The corresponding cylindrical profiles $B_z, B_\theta, q$ (right).*

Matlab packages were employed for unstructured triangular mesh generation and solving the equation (1). The same mesh was used for the stability calculations.

The necessary changes in the MHD_NX code [4] were implemented to deal with the RFP configurations. It concerns the regularity condition at the magnetic axis and periodicity conditions in $z$. For the perturbed electric field $\vec{e}$ we set the boundary condition on axis as $\vec{e} = 0$. There are two options to set the periodicity conditions: either set $\mathbf{e} \cdot \nabla\theta = 0$ at the periodicity ends (due to the ideal MHD condition $\mathbf{e} \cdot \mathbf{B} = 0$ and $\mathbf{B} \cdot \nabla r = 0$ it gives also $\mathbf{e} \cdot \nabla z = 0$ there) or directly apply the periodicity on the unknowns in the nodes and edges at the ends of the computational domain using additional Lagrange multipliers (provided that node positions in $r$ are identical at both ends). In the first case symmetric or antisymmetric in $z$ solutions should be identified to get the periodicity. Note that for the $m > 0$ case, when the complex amplitude of the electric field $\mathbf{e}_m e^{im\theta}$ is solved for, the real and imaginary parts of $\mathbf{e}_m$ should be symmetric or antisymmetric simultaneously. That is why the direct periodicity setting is more convenient and preferential for the stability analysis.

**2 Ideal MHD stability with islands** Low-q RFP configurations are unstable against free boundary $m = 1$ mode if conducting wall stabilization is not taken into account. However in RFP experiments the plasma is close to the wall and the corresponding RWM modes are actively controlled. The external $m = 1$ mode stability is hardly affected by the presence of islands (Fig.2). In both cylindrical and 2D island cases the squared growth rate $\omega^2/\omega_A^2 = -0.64$ with the conducting wall at $a_w/a = 2$ (the wall shape is similar to the plasma boundary), where Alfvén frequency is defined as $\omega_A^2 = (\psi_{max} - \psi_{min})^2/(a^6\rho)$ . The wall position for marginal stability of the $m/n = 1/1$ is about $a_w/a = 1.7$. Note that the global $m/n = 1/1$ mode is stable for the same equilibrium with $p = 2$ periods despite $q_a = -0.5$ in this case.

In the proposed 2D formulation the longitudinal "toroidal" harmonics $e^{inz/(pz_p)}$ are coupled unless the equilibrium is cylindrically symmetric. In the spectrum of the 2D eigenvalue problem for a given poloidal wave number $m = 1$ along with the global $m/n = 1/1$ unstable mode there are unstable modes with different $n$'s depending on the position of the corresponding resonant magnetic surface near the plasma boundary. The global $m/n = 1/1$ mode seems to be decoupled from the higher-n modes also in the case with islands, but for higher-n modes the coupling is more pronounced. However the growth rates of these modes localized near the plasma boundary are about ten times lower than for the global modes.

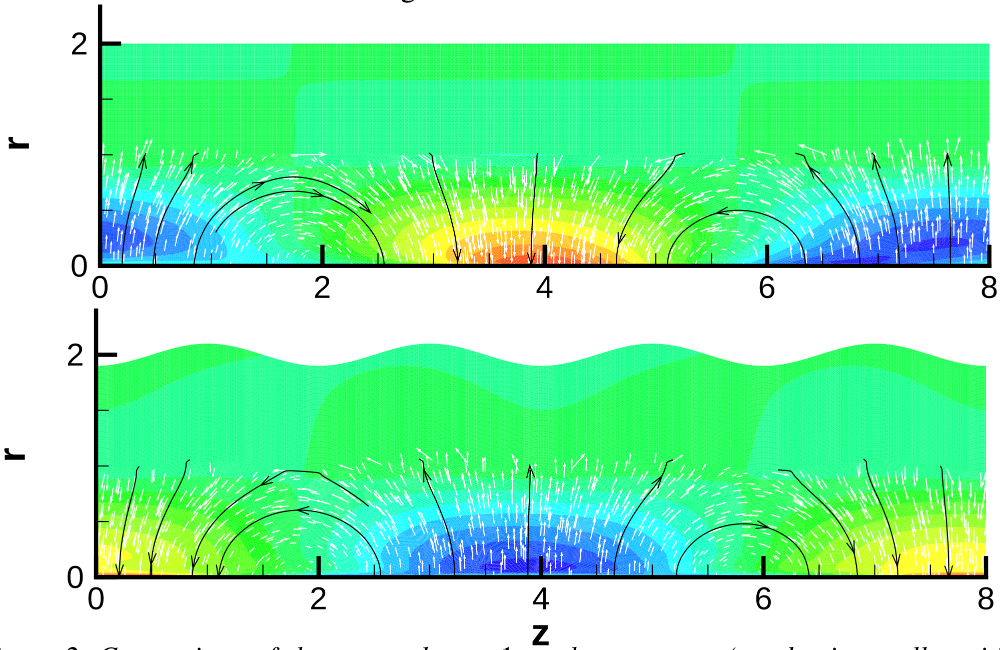


*Figure 2. Comparison of the external $m = 1$ mode structures (conducting wall position at $a_w/a = 2$) for the equilibrium from Fig.1 but extended to $p = 4$ periods (lower) and the cylindrical case (upper), $\lambda = 2.7$, $0.58 \geq q \geq -0.25$. The streamlines and arrow plots of plasma displacement $\xi$ projection onto $(r,z)$ plane and contour plot of $r\mathbf{e}\cdot\nabla\theta$ (in plasma equals to $\xi\cdot\nabla\psi/r$ related to the normal displacement to magnetic surfaces) are shown.*

The existence of islands brings a new $m = 0$ free-boundary instability: neither field reversal in a cylinder nor the boundary distortion without reversal give the instability. The mode structure of the external $m = 0$ island mode is shown in Fig.3 for the $p = 2$ period case. This is the most unstable mode which is localized in the vicinity of island X-points with dominating $n = p/2 = 1$ ($\omega^2/\omega_A^2 = -0.64$). For the considered shallow reversal case there is also the second unstable mode with dominating $n = p = 2$ ($\omega^2/\omega_A^2 = -0.36$). The same modes with the same growth rates are reproduced for $p = 4$ periods implying independence on the value of $q$.

The growth rate of the $m = 0$ island mode is not sensitive to the value of the safety factor $q$ but depends on the position of the field reversal with maximal growth rates taking place for shallow reversal with the islands closer to the boundary. The squared growth rate scales approximately linearly with the size of islands determined by the amplitude of the 2D boundary perturbation. For the considered case with $\lambda = 2.7$ the growth rate of the $m = 0$ mode is comparable to that of the global $m = 1$ mode with the wall at $a_w/a = 2$. However the marginal position of the conducting wall for the $m = 0$ island mode $a_w/a < 1.2$ is significantly closer to plasma. In Fig.4 the most unstable $m = 0$ island modes are presented for two cases with deeper field reversal.

**3 Discussion** The appearance of the new $m = 0$ free boundary instability was demonstrated in the 2D model of RFP configuration with $m = 0$ islands. The growth rate of this instability is comparable to that for external $m = 1$ mode in the low-q RFP configuration but in contrast to $m > 0$ modes does not depend on the value of safety factor. The growth rate increases and the marginal wall position of the $m = 0$ island mode decreases with shear reversal position closer to the plasma boundary. More accurate numerical treatment would be needed to investigate the cases with very shallow reversal $2.4 < \lambda a < 2.7$ which are potentially the most unstable configurations.

The relevance of the obtained results to the intrinsically 3D helical reversed field pinch configurations with $m = 0$ islands [5] (with characteristic island size about half of that in the presented 2D cases) needs to be assessed.

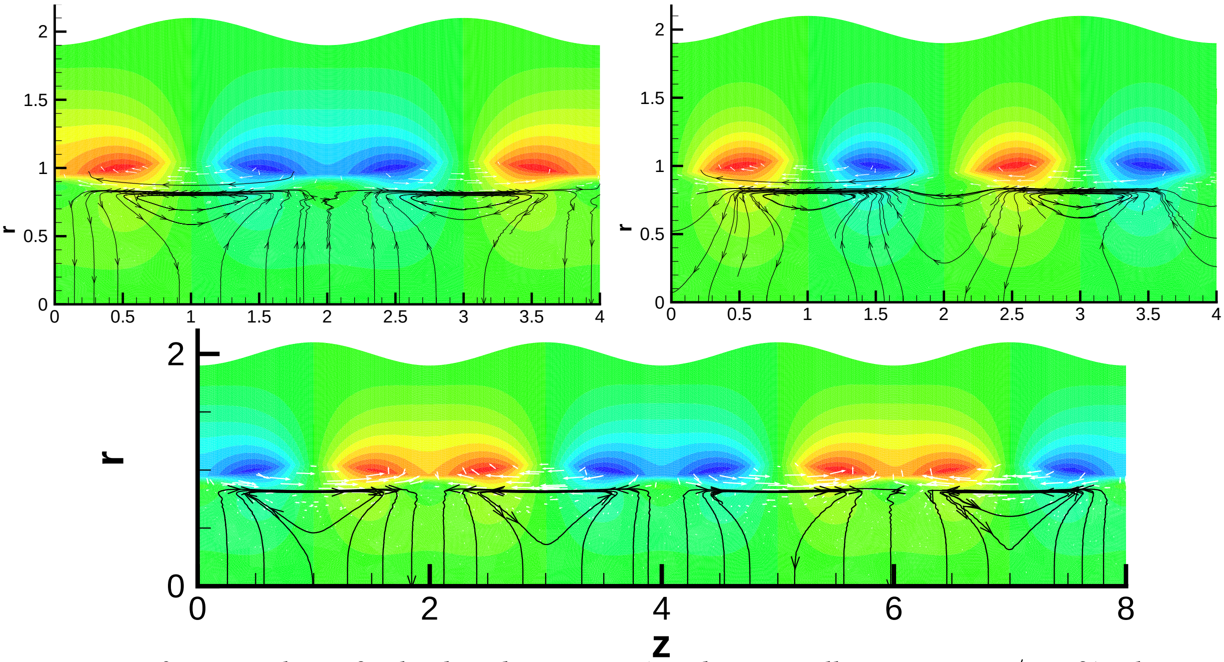


*Figure 3. External $m = 0$ island mode structure (conducting wall position at $a_w/a = 2$). The equilibrium from Fig.1 with $p = 2$ periods: the most unstable mode with $n = p/2$ and $\omega^2/\omega_A^2 = -0.64$ (upper left), the second unstable mode with $n = p$ and $\omega/\omega_A^2 = -0.36$ (upper right). The most unstable mode with $n = p/2$ and $\omega^2/\omega_A^2 = -0.64$ with $p = 4$ periods (lower).*

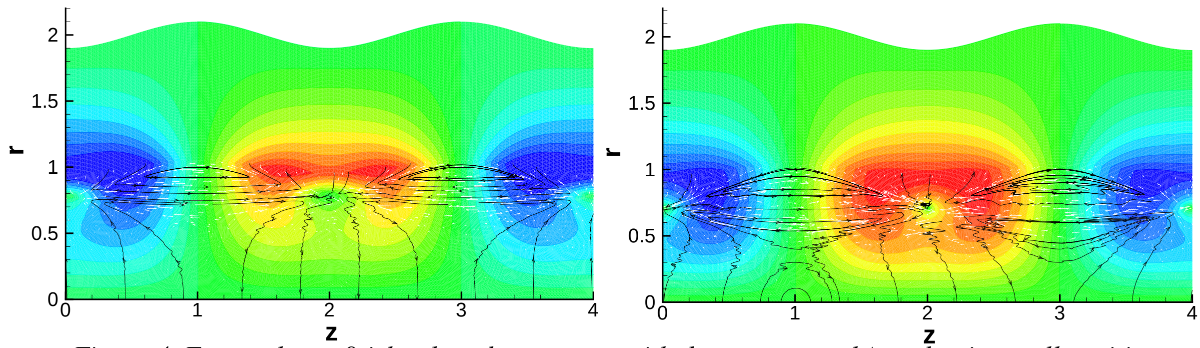


*Figure 4. External $m = 0$ island mode structure with deeper reversal (conducting wall position at $a_w/a = 2$). The equilibria with $p = 2$ periods: the most unstable mode with $n = p/2$ and $\omega^2/\omega_A^2 = -0.19$ for $\lambda = 3.0$ (left), the most unstable mode with $n = p/2$ and $\omega/\omega_A^2 = -0.05$ for $\lambda = 3.3$ (right).*

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