Stability of the pre-sheath in the Tonks-Langmuir discharge

D. D. Tskhakaya (sr.)[1,2], L. Kos[3], and S. Kuhn[1]

[1] Association EURATOM-ÖAW, Inst. for Theoretical Physics, Univ. of Innsbruck, Austria

[2] Andronikashvili Institute of Physics, Georgian Academy of Sciences, Tbilisi, Georgia

[3] Association EURATOM-SFA, Faculty of Mech. Eng., University of Ljubljana, Slovenia

Abstract

It is shown that regarding high-frequency perturbations the pre-sheath is stable. The stability problem for low-frequency perturbations can be reduced to analysing a "diffusion-type" equation, with the result that the potential distribution is weakly unstable.

1. In the papers on weakly ionized plasma-wall transition (PWT) layers (plasma boundary layers) published until now, only the time-independent ($\partial/\partial t = 0$) states of these layers have been studied while, to the best of our knowledge, their stability properties have been left out of consideration. Analytic description of a PWT layer as a whole single unit is quite unwieldy due to the mathematical difficulties involved. Therefore, a PWT layer is usually split into two sub-layers: a quasi-neutral, collisional pre-sheath (PS) (with the characteristic scale length l, the smallest relevant collision-related length) and the Debye sheath (DS) (with the scale length λ_D, the electron Debye length at the sheath entrance). This subdivision ensues naturally in the "asymptotic two-scale (a2s) limit" $\varepsilon \equiv \lambda_D / l \to 0$ [1] and allows one to investigate some aspects of the PS and the DS separately from each other.

Here, in particular, we consider the stability problem for the one-dimensional, symmetric Tonks-Langmuir (T&L) discharge model (see Fig. 1), for which the analytic solutions of the time-independent states are well-known [1]. The plasma, consisting of Boltzmann-distributed electrons with constant temperature T_e and singly charged ions, is bounded on both sides by absorbing walls located at $z = \pm L$, and biased negatively at floating potential. The ion kinetics is governed by electron-impact ionization of a cold, uniformly distributed neutral-gas background. With the dimensionless potential, velocity, time, distribution function and densities

$$\overline{\varphi} = -e\phi/T_e \; (v/c_s) \to v \,, \;\; \omega_{pi} t \to t \,, \;\; \left(c_s/\sqrt{2}n_0\right) f_i = \bar{f} \,, \;\; (v_i/c_s) = 1/l \,, \;\; \left(n_{i,e}/n_0\right) \Rightarrow n_{i,e} \,,$$

the ion Boltzmann equation and Poisson's equation can be written in the form

$$\frac{1}{\lambda_D}\frac{\partial \bar{f}}{\partial t}+v\frac{\partial \bar{f}}{\partial z}+\frac{\partial \bar{\varphi}}{\partial z}\frac{\partial \bar{f}}{\partial v}=\frac{1}{\sqrt{2}l}e^{-\bar{\varphi}}\delta(v), \qquad \lambda_D^2\frac{\partial^2 \varphi}{\partial z^2}=n_i-n_e. \tag{1a,b}$$

Here, $\lambda_D=\sqrt{\varepsilon_0 T_e/(e^2 n_0)}$, $c_s=\sqrt{T_e/m_i}$, n_0 is the density in the centre of the discharge, ν_i is the "ionization frequency" (the average number of ionizations an electron causes per second), $l\equiv c_s/\nu_i$ is the "ionization length", $\delta(v)$ is the Dirac δ function (representing the velocity distribution of the immobile neutrals), and. Following the common procedure of stability analysis, we decompose the distribution function (DF) $\bar{f}(t,z,v)$ and the electric potential $\bar{\varphi}(t,z)$ into their unperturbed values and time-dependent perturbations,

$$\bar{f}=f(z,v)+\delta f(t,z,v), \qquad \bar{\varphi}=\varphi(z)+\delta\varphi(t,z), \qquad (\delta f<<f,\quad \delta\varphi<<\varphi), \tag{2a,b}$$

where $f(z,v)$ and $\varphi(z)$ obviously describe the unperturbed time-independent state.

2. For describing the time-independent PS, we introduce the dimensionless coordinate $x=z/l$ ("PS scale"), with which Eqs. (1a,b) become

$$v\frac{\partial f}{\partial x}+\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v}=\frac{1}{\sqrt{2}}e^{-\varphi}\delta(v), \quad \varepsilon^2\frac{\partial^2\varphi}{\partial x^2}=n_i-n_e \tag{3a,b}$$

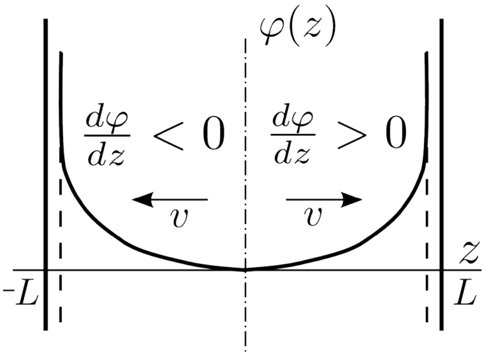

Fig.1. Geometry of the system.

In the PS the parameter $\varepsilon\equiv\lambda_D/l$ is assumed to be small and the a2s limit $\varepsilon\to 0$ allows us to consider the PS as an independent quasi-neutral region [1]. Solving Eq. (3a) first, we have to distinguish the cases with positive ($v>0$) and negative ($v<0$) velocities and consequently consider these cases separately in the usual coordinate space, $x<0$ and $x<0$ (see Fig. 1). With the quasi-neutrality condition $n_i=n_e$ we have from (3a) [1]

$$f(\varphi,v)=\frac{1}{\pi}\frac{\partial}{\partial\varphi}F\left(\sqrt{\varphi-\frac{v^2}{2}}\right)\cdot H\left(\varphi-\frac{v^2}{2}\right), \quad \text{with} \quad F\left(\sqrt{s}\right)=e^{-s}\int_0^{\sqrt{s}}dt\cdot e^{t^2}, \tag{4}$$

with $H(s)$ the Heaviside step function. The Poisson equation containing the perturbations, the characteristic scale length of which is assumed to be λ_D, has the form

$$\frac{\partial^2}{\partial\xi^2}\delta\varphi=\sqrt{2}\int_{-\infty}^{\infty}dv\cdot\delta f+\delta n_e(\xi.t), \tag{5}$$

with $\xi = z/\lambda_D$ ("DS scale"). Neglecting the terms of the order of the small ratio λ_D / l and requiring the perturbations to vanish at $t = t_0 \to -\infty$, we obtain for the ion-DF perturbation

$$\delta f = v \cdot f'\left(\varphi - v^2/2\right) \cdot \frac{\partial}{\partial \xi} \int_{t_0 \to -\infty}^{t} dt' \cdot \delta\varphi\{\xi - v(t-t'), t'\}, \quad \text{where } f'(s) = df(s)/ds . \qquad (6)$$

3. In our case we can apply methods of geometrical optics because the scale length of the perturbations is much smaller than that of the unperturbed medium, $\lambda_D << l$. With the Fourier transformation $\delta\varphi(\xi, t) \Rightarrow \varphi(k,\omega) \cdot e^{ik\xi - i\omega t}$, Eqs. (5) and (6) lead to the eikonal equation

$$k^2(\omega, x) = -i\pi\sqrt{2}\frac{\omega}{k} f'\left(\varphi(x) - \omega^2/2k^2\right) + \sqrt{2} P \int_{-\infty}^{\infty} dv \frac{v f'\left(\varphi(x) - v^2/2\right)}{(\omega/k) - v} - \qquad (7)$$

$$- e^{-\varphi(x)}\left\{1 - \Im_+\left((\omega/k)\sqrt{m_e/m_i}\right)\right\}; \qquad \Im_+(s) = \frac{1}{\sqrt{2\pi}} s \int_{-\infty}^{\infty} \frac{e^{-z^2/2} dz}{s + i\delta - z}, \qquad \delta \to +0 ,$$

where the operator P prescribes that the principal value is to be taken. The solutions ω of this equation are the eigenfrequencies of the system and are in general complex ($\omega \Rightarrow \omega + i\gamma$, $\gamma << \omega$), with the real part ω the oscillation frequency and the imaginary part γ the growth rate of the perturbations. These real and imaginary parts are found from [2, 3]

$$\int_{x_1}^{x_2} dx \cdot \mathrm{Re}\, k(\omega, x) = N\pi , \ N >> 1, \qquad \gamma = -\left\{\int_{x_1}^{x_2} dx\, \mathrm{Im}\, k(\omega, x)\right\} \cdot \left\{\int_{x_1}^{x_2} dx \frac{\partial}{\partial \omega} \mathrm{Re}\, k(\omega, x)\right\}^{-1} \quad (8a,b)$$

4. (i) In the high-frequency range ($\omega^2/k^2 >> 2\varphi$), the simplified dispersion relation (7) is similar to the relation given in [2] (see Eq. (8.49) there). Straightforward calculations show that $\gamma < 0$ so the high-frequency perturbations are damped

(ii) Applying the eikonal and quantization equations (7) and **(8)** to low-frequency perturbations ($\omega/k << 1$) we find

$$(\mathrm{Re}\, k)^4 = e^{-\varphi}\omega^2, \quad \mathrm{Im}\, k = -\frac{\omega \cdot e^{-\varphi(x)}}{2(\mathrm{Im}\, k)^3}\left\{\frac{\pi}{\sqrt{2}} e^{-\varphi(x)} (\mathrm{Re}\, k) \cdot f'\left(\varphi(x) - \frac{v^2}{2}\right) + \frac{1}{2}\sqrt{\frac{\pi}{2}} (\mathrm{Re}\, k) \sqrt{\frac{m_e}{m_i}}\right\}. \quad (9a,b)$$

From these relations we conclude: (a) In the low-frequency range the system has no eigenfrequencies and hence cannot be described in the framework of common geometrical optics; (b) Equation (9b) indicates that, according to (6), at phase velocities close to $\sqrt{2\varphi}$, $\omega/\mathrm{Re}\, k \approx \sqrt{2\varphi}$, the imaginary part of the frequency can become positive, $\gamma > 0$, and even tend to the infinity, $\gamma \to \infty$, so the perturbations can be unstable.

5. In the limit of smooth time dependence, when $\left(\partial^2\delta\varphi/\partial t^2\right) << \left(\partial^2\delta\varphi/\partial\xi^2\right)$, Eqs. (5)-(6) can be reduced to the "diffusion -like" form

$$\frac{\partial^4\delta\varphi}{\partial\xi^4} = -\frac{\partial^2\delta\varphi}{\partial t^2}, \tag{10}$$

which is in accordance with Eq. (9a). The solution of (10) is

$$\delta\varphi(\xi,t) = \frac{1}{2\sqrt{2\pi\cdot t}}\int_{-\infty}^{\infty} d\xi'\delta\varphi(\xi',0)\cdot\left\{\cos\frac{(\xi-\xi')^2}{4t} + Sin\frac{(\xi-\xi')^2}{4t}\right\} \tag{11}$$

$$+\sqrt{\frac{t}{2\pi}}\int_{-\infty}^{\infty} d\xi'\delta\dot{\varphi}(\xi',0)\cdot\left\{Cos\frac{(\xi-\xi')^2}{4t} + Sin\frac{(\xi-\xi')^2}{4t} + \frac{|\xi-\xi'|}{2\sqrt{t}}\sqrt{2\pi}\left[S\left(\frac{\xi-\xi'}{2\sqrt{t}}\right) - C\left(\frac{\xi-\xi'}{2\sqrt{t}}\right)\right]\right\},$$

where $\delta\varphi(\xi,0) = \delta\varphi(\xi,t)\big|_{t=0}$, $\delta\dot{\varphi}(\xi,0) = \left(\partial\delta\varphi(\xi,t)/\partial t\right)\big|_{t=0}$, and $S(s) = \sqrt{\frac{2}{\pi}}\int_0^s ds\cdot Sin(s^2)$, $C(s) = \sqrt{\frac{2}{\pi}}\int_0^s ds\cdot Cos(s^2)$ are the Fresnel integrals. To simplify the analysis we consider two particular cases: **(a)** At $t = 0$ the potential perturbation is localized in a narrow region and its time derivative there is zero, $\delta\varphi(\xi,0) = a\cdot\delta(\xi-\xi_0)$ and $\delta\dot{\varphi}(\xi,0) = 0$. From Eq. (11) we find

$$\delta\varphi(\xi,t) = \frac{a}{2\sqrt{\pi\cdot t}}\left\{\cos\frac{(\xi-\xi_0)^2}{4t} - \frac{\pi}{4}\right\}, \qquad t > 0. \tag{12}$$

Hence, the potential perturbation executes oscillations with decreasing amplitude. **(b)** Quite different behaviour is observed if at $t = 0$ the potential suffers a blow at some definite point, $\delta\dot{\varphi}(\xi,0) = \bar{a}\cdot\delta(\xi-\xi_0)$, whereas the initial potential perturbation itself is zero, $\delta\varphi(\xi,0) = 0$. From (11) we then have

$$\delta\varphi(\xi,t) = \bar{a}\sqrt{\frac{t}{2\pi}}\left\{\cos\frac{(\xi-\xi_0)^2}{4t} + Sin\frac{(\xi-\xi_0)^2}{4t}\right\} + \frac{|\xi-\xi_0|}{2\sqrt{t}}\sqrt{2\pi}\left\{S\left(\frac{\xi-\xi_0}{2\sqrt{t}}\right) - C\left(\frac{\xi-\xi_0}{2\sqrt{t}}\right)\right\}. \tag{13}$$

Hence, the amplitude of the perturbation grows proportionally to $\sqrt{t}$.

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