Three-dimensional magnetized and rotating hot plasma equilibrium and stability in a gravitational field

Peter J. Catto[1], Sergei I. Krasheninnikov[2], Istvan Pusztai[3]

[1] *Plasma Science and Fusion Center, MIT, Cambridge, MA 02139, USA*

[2] *University of California at San Diego, La Jolla, CA 92093, USA*

[3] *Department of Applied Physics, Chalmers U. of Technology, 41296 Gothenburg, Sweden*

Abstract

We present analytic solutions for three-dimensional magnetized axisymmetric equilibria confining rotating hot plasma in a gravitational field. Our solution to the full Grad-Shafranov equation can exhibit strong equatorial plane localization of the plasma density and current, resulting in disk equilibria for the plasma density. Unlike in [1], we find a toriodal magnetic field is necessary to find a equilibrium in the presence of gravity for most cases of interest. We expect our results to provide impetus to re-investigate magneto-rotational stability [2,3].

Introduction: Fully self-consistent three dimensional global equilibrium of hot, rotating plasma confined by gravity and magnetic field are of interest for astrophysical and space plasma applications, but have proven difficult to find. Previous magnetohydrodynamic (MHD) models assume strict incompressibility with constant density [4], assume the density is a flux function [5], ignore the frozen in constraint [6], and/or require poloidal flow [5-7] or an adiabatic equation of state [8,9], which are not allowed kinetically. To satisfy constraints imposed on a drifting Maxwellian by the Fokker-Planck equation, only toroidal flow is allowed and the temperature must be a flux function [1,10-12]. Moreover, in the presence of rotation and gravity the density must be allowed to vary poloidally as well as radially since strict Keplerian motion is only possible at the equatorial plane [1]. It is anticipated that the equilibria we find will be useful in setting up global simulations to investigate magneto-rotational stability [2,3] in accretion disks and help shed light on the detailed mechanisms by which mass accretion occurs at a black hole as momentum is transported outward.

Grad-Shafranov equation: The flux surfaces for an axisymmetric equilibrium must satisfy a Grad-Shafranov equation that we find from Ampere's law and pressure balance by taking the magnetic field to be given by

$$\vec{B} = I\nabla\zeta + \nabla\psi\times\nabla\zeta , \tag{1}$$

where ζ is the toroidal angle, ψ is the poloidal flux function, $I = RB_T$ with B_T the toroidal magnetic field and R the cylindrical radius from the axis of symmetry. To satisfy the kinetic constraints [10-12] the velocity $\vec{V}$ must be toroidal

$$\vec{V} = \Omega R^2 \nabla\zeta , \tag{2}$$

with $\Omega = c\, d\Phi/d\psi$ the toroidal rotation frequency. The magnetic field is frozen into the flow so that $c\nabla\Phi = \vec{V}\times\vec{B}$ gives $\vec{B}\cdot\nabla\Phi = 0$, requiring the electrostatic potential Φ to be a flux function to lowest order. In addition, we write the gravitational potential G as

$$G = -G_O M_O / r \tag{3}$$

with G_O the gravitational constant, r the spherical radius, and M_O the mass of the astrophysical body that is assumed to be a compact source centered at r = 0, such that R = r sinϑ with ϑ the angle from the axis of symmetry. Our spherical and cylindrical coordinates are defined to satisfy $r\nabla\vartheta = R\nabla\zeta\times\nabla r$ and $\nabla z = R\nabla R\times\nabla\zeta$.

Conservation of momentum requires

$$c^{-1}\vec{J}\times\vec{B} = Mn(\nabla G - \Omega^2 R\nabla R) + \nabla p . \tag{4}$$

The parallel component requires that the ion density n depend on poloidal angle

$$n = n(\psi,\vartheta) = \eta(\psi)e^{\kappa(\psi,\vartheta)} = \eta(\psi)e^{\kappa(\psi,\vartheta)} , \tag{5}$$

where the density and normalizing or "pseudo" density η are related via the generalized Maxwell-Boltzmann exponential factor κ that retains the poloidal dependence due to the centrifugal and gravitational potential,

$$4T\kappa/M = \Omega^2 R^2 - 2G , \tag{6}$$

with T = T(ψ) and p = n(T_i+ZT_e) = 2nT in a quasi-neutral plasma (Zn = n_e = electron density) of ion charge number Z and ion and electron temperatures T_i and T_e. The toroidal momentum balance requires the current density, $\vec{J}$, across a flux surface vanish, $\vec{J}\cdot\nabla\psi = 0$, and then Ampere's law gives I = I(ψ). Ampere's law also gives $4\pi c^{-1}\vec{J}\cdot\nabla\vartheta = \vec{B}\cdot\nabla\vartheta dI/d\psi$. Then the current density can be conveniently decomposed into toroidal and parallel components by writing it as $\vec{J} = RJ_*\nabla\zeta + (c/4\pi)dI/d\psi\vec{B}$, allowing us to obtain the useful relation $\vec{J}\cdot\vec{B} = I\vec{J}\cdot\nabla\zeta + (cB_P^2/4\pi R^2)dI/d\psi$, with $\vec{B}_P = \nabla\psi\times\nabla\zeta$ the poloidal magnetic field and $B_P^2 = R^{-2}|\nabla\psi|^2$. Combining this result with the $\nabla\psi$ component of force balance, $R^2B^2\vec{J}\cdot\nabla\zeta = I\vec{J}\cdot\vec{B} + c\nabla\psi\cdot[\nabla p + Mn(\nabla G - \Omega^2 R\nabla R)]$, and the toroidal component of Ampere's law yields the Grad-Shafranov equation [13]

$$\nabla\cdot(R^{-2}\nabla\psi) = -IR^{-2}dI/d\psi - 4\pi R^{-2}B_P^2\nabla\psi\cdot[\nabla p + Mn(\nabla G - \Omega^2 R\nabla R)] . \tag{7}$$

Separable form: To find a separable form for the Grad-Shafranov equation we assume

$$\psi = \psi_O H(\mu)(R_O/r)^{\alpha} , \tag{8}$$

where $\mu = \cos\vartheta$ and our normalization is $H(\mu=0) = 1$. The vacuum limit $H = 1 - \mu^2$ recovers a homogeneous magnetic field for $\alpha = -2$ and a point dipole solution if $\alpha = 1$, and ψ_O is a constant reference value at the reference location R_O (reference values are defined at the equatorial plane $\mu = 0$ and denoted by a subscript "o"). We obtain an ordinary differential equation for H by assuming

$$\Omega^2 = \Omega_O^2(\psi/\psi_O)^{3/\alpha} \propto 1/r^3 , \tag{9}$$

$$T = T_O(\psi/\psi_O)^{1/\alpha} , \tag{10}$$

$$I = I_O(\psi/\psi_O)^{1+1/\alpha} , \tag{11}$$

$$n = n_O(\psi/\psi_O)^{2+3/\alpha} e^{\kappa(\mu)} , \tag{12}$$

with $n_O = n(\psi = \psi_O, \mu = 0)$. Defining the positive constants

$$g = \frac{8\pi G_O M_O M n_O}{B_{PO}^2 R_O} = \left.\frac{8\pi G_O M_O M n}{B_P^2 r}\right|_{\mu=0} , \tag{13}$$

$$\omega^2 = \frac{4\pi M n_O \Omega_O^2 R_O^2}{B_{PO}^2} = \left.\frac{4\pi M n \Omega^2 r^2}{B_P^2}\right|_{\mu=0} , \tag{14}$$

$$\beta = \frac{16\pi n_O T_O}{B_{PO}^2} = \left.\frac{8\pi p}{B_P^2}\right|_{\mu=0} , \tag{15}$$

and

$$b^2 = \frac{I_O^2}{R_O^2 B_{PO}^2} = \left.\frac{I^2}{R^2 B_P^2}\right|_{\mu=0} = \left.\frac{B_T^2}{B_P^2}\right|_{\mu=0} , \tag{16}$$

an ordinary second order nonlinear differential equation for H is obtained from (7) [13]:

$$\frac{d^2H}{d\mu^2} + \frac{\alpha(\alpha+1)}{1-\mu^2}H = \alpha[\frac{g}{2}H^{-1/\alpha} - \omega^2(1-\mu^2)H^{2/\alpha} - (\alpha+2)\beta - \frac{(\alpha+1)b^2}{(1-\mu^2)H^{2/\alpha}}]H^{1+4/\alpha}e^{\kappa} , \tag{17}$$

where

$$\kappa = -(g/\beta)(1 - H^{-1/\alpha}) - (\omega^2/\beta)[1 - (1-\mu^2)H^{2/\alpha}] , \tag{18}$$

Solution technique: Solutions to this Grad-Shafanov equation can be found by the techniques illustrated in [1] and references therein, however, some care is needed. The solution H must be up-down symmetric and monotonically decreasing from unity until the poloidal magnetic field vanishes at the axis of symmetry ($1 \geq H \geq 0$). This behavior seems to require $d^2H/d\mu^2 < 0$ for all μ. Consequently, we desire $g - (\alpha+1)(2b^2+1) > 2\omega^2 + 2(\alpha+2)\beta$ at $\mu = 0$. Moreover, gravity forces us to assume $\alpha < 0$ to avoid singular behavior as $H \to 0$, with rotation requiring $H \leq (1-\mu^2)^{-\alpha/2}$ to avoid exponential growth at the poles. An exact solution,

$H=(1-\mu^2)^{-\alpha/2}$, exists if G = 0 [1] when $\alpha=-[2(\beta+1)+\omega^2+b^2]/(\beta+1+b^2)$. The poloidal magnetic field decreases with radius if $\alpha > -2$ and the toroidal magnetic field remains finite at the poles if $\alpha < -1$ so we expect solutions with $-1 > \alpha \geq -2$ are of most interest in the presence of gravity. The g = 0 solution suggests maintaining $d^2H/d\mu^2 < 0$ as $\mu^2 \to 1$ requires $[-(\alpha+1)]b^2 > \omega^2 + (\alpha+2)\beta$, a condition that is needed even when g is retained since $H^{-1/\alpha}\to 0$. This restriction implies a toroidal magnetic field is needed to obtain a fully self-consistent gravitational equilibrium. However, the procedures of [1] remain valid for obtaining solutions for $-1 > \alpha \geq -2$ with b^2 retained. Consequently, it is possible to find plasma disk equilibrium solutions to the Grad-Shafranov equation (17) and (18) that predict the rotation frequency bound, and the conditions for a plasma disk as well as its thickness.

Magneto-rotational instability (MRI) without gravity: In the absence of gravity the toroidal magnetic field can be neglected. In this case $\alpha < -2$, but the MRI calculation is more tractable [2]. The equilibrium associated with our exact solution for H is unstable if $\beta >> 1$ and $\Omega^2R^2 < 10T/M$ in the incompressible limit (roughly in agreement with [2]), and when $\beta < 3/2$ and $\Omega^2R^2 >> 2T/M$ in the compressible limit not considered in [2]. In the presence of gravity [3] the toroidal magnetic field must be retained to investigate MRI instability.

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