Toroidal rotation in resonant regimes of tokamak plasmas due to non-axisymmetric perturbations in the action-angle formalism

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Introduction

Toroidal torque generated via neoclassical toroidal viscosity (NTV) [1, 2, 3, 4] caused by external non-resonant, non-axisymmetric magnetic perturbations (TF-ripples, error fields, perturbations caused by ELM mitigation coils away from resonant surfaces) has a significant effect on toroidal plasma rotation in tokamaks. Besides collisional transport regimes, an important role (in particular, in ASDEX-Upgrade [4]) is played by resonant transport regimes such as superbanana-plateau [1] or bounce and bounce-transit resonance [2] regimes where transport coefficients are independent of the (small) collision frequency. Here, a universal approach by the canonical Hamiltonian quasilinear formalism in action-angle variables [5] is presented. This approach is well developed for the general case of small electromagnetic perturbations in tokamaks (see e.g. [6]). The described treatment covers both trapped and passing orbits in a unified way and does not require any simplifications of device geometry. Numerical results of NTV evaluation in resonant transport regimes are presented and compared to references [1, 3].

Toroidal torque and radial transport within canonical Hamiltonian quasilinear theory

Within the action-angle formalism, the quasilinear kinetic equation describing the evolution of the slowly varying averaged part of the distribution function $f_0 = f_0(\mathbf{J})$ over the canonical angles can be cast to the form

$$\frac{\partial f_0}{\partial t} = \sum_{\mathbf{m}} m_i \frac{\partial}{\partial J_i} Q_{\mathbf{m}}, \qquad Q_{\mathbf{m}} = Q_{\mathbf{m}}(\mathbf{J}) = \frac{\pi}{2} |H_{\mathbf{m}}|^2 \, \delta\left(m_j \Omega^j - \omega\right) m_k \frac{\partial f_0}{\partial J_k}, \tag{1}$$

where J_k are the actions, ω is the perturbation frequency, $\delta(\dots)$ is the Dirac delta-function. Summation is assumed over repeated indices i, j and k, and bold typesetting denotes the full set of three quantities, $\mathbf{J} = (J_1, J_2, J_3)$. Splitting $H = H_0 + \tilde{H}$ of the Hamiltonian H into an unperturbed (averaged over angles) part $H_0 = H_0(\mathbf{J})$ and a perturbation part $\tilde{H}$ leads to the following definitions of the canonical frequencies Ω^j and of the amplitudes $H_{\mathbf{m}} = H_{\mathbf{m}}(\mathbf{J})$ of the Fourier expansion of $\tilde{H}$ over the canonical angles θ^k:

$$\Omega^i = \frac{\partial H_0}{\partial J_i}, \qquad \tilde{H} = \mathrm{Re} \sum_{\mathbf{m}} H_{\mathbf{m}} \mathrm{e}^{i\left(m_k \theta^k - \omega t\right)}, \tag{2}$$

where summation is performed over all three Fourier indices $\mathbf{m} = (m_1, m_2, m_3)$. In a tokamak, two of the actions are the perpendicular adiabatic invariant (up to a constant factor the magnetic moment) $J_1 = J_\perp$ and the canonical toroidal momentum $J_3 = p_\varphi$ respectively given by

$$J_\perp \approx \frac{m_\alpha v_\perp^2}{2\omega_{c\alpha}}, \qquad p_\varphi = m_\alpha v_\varphi + \frac{e_\alpha}{c} A_\varphi \approx m_\alpha v_\parallel h_\varphi + \frac{e_\alpha}{c} A_\varphi, \tag{3}$$

where m_α, e_α and $\omega_{c\alpha}$ are α-species mass, charge and cyclotron frequency, respectively, v_φ, h_φ and A_φ stand for the toroidal co-variant components of particle velocity, unit vector along the magnetic field and vector potential, respectively, and $v_\perp$ and $v_\parallel$ are perpendicular and parallel velocities (all quantities are defined for the unperturbed magnetic field). In flux (Boozer)

coordinates (s,ϑ,φ) where $s=\psi_{\rm tor}/\psi^a_{\rm tor}$ is the toroidal flux normalized by its edge value, $A_\varphi=-\psi_{\rm pol}(s)$ is a flux function (poloidal flux). Introducing $s_\varphi=s_\varphi(p_\varphi)$, which is the solution to the second equation in (3) with $v_\parallel$ set to zero (banana tips in case of trapped orbits), the poloidal action $J_2=J_\vartheta$ is defined for both, trapped and passing particles by

$$J_\vartheta=\frac{e_\alpha}{c}A_\vartheta(s_\varphi)\delta_{t-p}+J_\parallel,\qquad \text{trapped: } \delta_{t-p}=0,\quad \text{passing: } \delta_{t-p}=1, \tag{4}$$

where $A_\vartheta(s)=s\psi^a_{\rm tor}$ and the parallel adiabatic invariant is defined via the bounce average as

$$J_\parallel=\frac{m_\alpha\tau_b}{2\pi}\langle v_\parallel^2\rangle_b,\qquad \langle a(\vartheta)\rangle_b\equiv\frac{1}{\tau_b}\int\limits_0^{\tau_b}\mathrm{d}\tau\, a\left(\vartheta_{\rm orb}(\vartheta_0,\tau)\right). \tag{5}$$

Here $a(\vartheta)$ is any function of poloidal angle and integrals of motion $(J_\perp,H_0,s_\varphi)$, $\vartheta_{\rm orb}(\vartheta_0,\tau)$ is the (periodic) solution of the unperturbed guiding center (orbit) equations starting at ϑ_0 ($B(\vartheta_0)=B_{\rm min}$), and τ_b is the bounce time, $\vartheta_{\rm orb}(\vartheta_0,\tau_b)=\vartheta_{\rm orb}(\vartheta_0,0)=\vartheta_0$. Since $\Omega^1=\langle\omega_{c\alpha}\rangle_b$ is much larger then the other frequencies, for quasi-static ($\omega\to0$) magnetic perturbations with perpendicular scale much larger than Larmor radius and banana width only $m_1=0$ contributes significantly in (1). Such Fourier amplitudes correspond to a gyroaverage because the gyrophase is a linear function of the first canonical angle, $\phi=\theta^1+\Delta\phi(\theta^2,\mathbf{J})$, while the remaining guiding center variables are independent of θ^1. For a similar reason the Fourier amplitudes of the Hamiltonian perturbation in the form of a single toroidal harmonic $\propto\exp(in\varphi)$ differ from zero only for $m_3=n$, since the toroidal angle φ is the only coordinate depending (linearly) on the canonical angle θ^3, as given by a variable transformation in linear order over the Larmor radius,

$$\theta^2=\Omega^2\tau,\qquad \varphi=\theta^3-q\theta^2\delta_{t-p}+q\vartheta_{\rm orb}(\vartheta_0,\tau). \tag{6}$$

$\Omega^2=\Omega^2(\mathbf{J})=\omega_b=2\pi\tau_b^{-1}$ is the bounce frequency and $q=q(s_\varphi)$ the safety factor. Namely, within ideal MHD quasistatic electromagnetic perturbations are fully described by perturbations of the B module on perturbed flux surfaces, $B=B_0(\vartheta)+\mathrm{Re}\left(B_n(\vartheta)\exp(in\varphi)\right)$, which results in

$$H_{\mathbf{m}}=\left\langle\left(m_\alpha v_\parallel^2(\vartheta)+J_\perp\omega_{c\alpha}(\vartheta)\right)\frac{B_n(\vartheta)}{B_0(\vartheta)}\mathrm{e}^{inq\vartheta-i\left(m_2+nq\delta_{t-p}\right)\omega_b\tau}\right\rangle_b, \tag{7}$$

where $\vartheta=\vartheta_{\rm orb}(\vartheta_0,\tau)$ in all functions of ϑ and only $\mathbf{m}=(0,m_2,n)$ with various m_2 contributing to the result. Thus, the resonance condition given by the δ-function in Eq. (1) is reduced to

$$m_j\Omega^j-\omega=0\quad\rightarrow\quad m_2\omega_b+n\Omega^3=0, \tag{8}$$

$$\Omega^3=q\omega_b\delta_{t-p}+\Omega_{\rm tor},\qquad \Omega_{\rm tor}\equiv-\omega_b\frac{\partial J_\parallel}{\partial p_\varphi}=\Omega_{tE}+\langle\Omega_{tB}\rangle_b,\qquad \Omega_{tE}=-\frac{cq}{\psi^a_{\rm tor}}\frac{\mathrm{d}\Phi}{\mathrm{d}s}, \tag{9}$$

with Φ being the equilibrium electrostatic potential. In Eq. (9) only the contribution of the $\mathbf{E}\times\mathbf{B}$ drift to the cross-field toroidal rotation frequency $\Omega_{\rm tor}$ is given explicitly because Ω_{tB} is rather complicated. Condition (8) describes all regimes of interest here: The resonance $m_2=0$ for trapped particles corresponds to the superbanana-plateau resonance, $m_2=m$ for passing with poloidal mode m gives a transit resonance, which is the only one surviving in the infinite aspect ratio limit where it reduces to a Cherenkov (TTMP) resonance. Finite mode numbers m_2 correspond to bounce and bounce-transit resonances for trapped and passing particles, respectively.

In the case of small enough perturbations considered here, quasilinear effects are weak so that f_0 is close to a local Maxwellian. Then Eq. (1) can be replaced by a set of radial transport equations for the moments of f_0. A transport equation of interest is the conservation law of generalized toroidal momentum, which is obtained by multiplying Eq. (1) by $p_\varphi\delta\left(s-s_{\rm can}(\theta,\mathbf{J})\right)$,

integrating the product over phase space and dividing the result by $\mathrm{d}V/\mathrm{d}s$,

$$\frac{\partial}{\partial t}\left\langle \int \mathrm{d}^3 p p_\varphi f_0 \right\rangle + \frac{\mathrm{d}s}{\mathrm{d}V}\frac{\partial}{\partial s}\left(\frac{\mathrm{d}V}{\mathrm{d}s}F_{p_\varphi}\right) = T_\varphi, \tag{10}$$

where s_{can} is a variable change law, V is the volume within the flux surface and $\langle \dots \rangle$ is the neoclassical "flux surface" average. The flux surface averaged toroidal torque density T_φ, which dominates the transport term containing the momentum flux density F_{p_φ}, is

$$T_\varphi = -n\frac{\mathrm{d}s}{\mathrm{d}V}\int \mathrm{d}^3\theta \int \mathrm{d}^3 J \delta(s - s_{\mathrm{can}}) \sum_{m_2} Q_{\mathbf{m}} \approx -2\pi^2 n m_\alpha^3 \frac{\mathrm{d}\psi_{\mathrm{pol}}}{\mathrm{d}V}\int_0^\infty \mathrm{d}v v^3 \int_0^{1/B_{\min}} \mathrm{d}\eta\, \tau_b \sum_{m_2} Q_{\mathbf{m}}, \tag{11}$$

where setting $s_{\mathrm{can}} \approx s_\varphi$ (ignoring the FLR effects) and using the δ-function made the integration over angles and J_3 trivial. The remaining integration variables were changed from $J_\perp$ and J_ϑ to $\eta = v_\perp^2/(v^2 B_0)$ and velocity module v. Substituting f_0 in (1) by a drifting Maxwellian, $f_0 = (2\pi m_\alpha T_\alpha)^{-3/2} n_\alpha \exp\left((e_\alpha \Phi - H_0)/T_\alpha\right)$, where $n_\alpha = n_\alpha(s_\varphi)$ as well as T_α and Φ, for $\mathbf{J}$ satisfying the resonance condition one obtains

$$m_k \frac{\partial f_0}{\partial J_k} = -\frac{n c f_0 \left(A_1 + A_2 u^2\right)}{e_\alpha \langle|\nabla \psi_{\mathrm{pol}}|\rangle}, \quad \frac{A_1}{\langle|\nabla s|\rangle} = \frac{1}{n_\alpha}\frac{\partial n_\alpha}{\partial s} + \frac{e_\alpha}{T_\alpha}\frac{\partial \Phi}{\partial s} - \frac{3}{2T_\alpha}\frac{\partial T_\alpha}{\partial s}, \quad \frac{A_2}{\langle|\nabla s|\rangle} = \frac{1}{T_\alpha}\frac{\partial T_\alpha}{\partial s},$$

where $u = v/v_T$ and $v_T = \sqrt{2T_\alpha/m_\alpha}$. Relating T_φ to the flux-surface averaged particle flux density $\Gamma = -n_\alpha(D_{11}A_1 + D_{12}A_2)$ via the flux-force relation [1] $T_\varphi = -e_\alpha c^{-1}\langle|\nabla\psi_{\mathrm{pol}}|\rangle\Gamma$, resonant transport coefficients follow as

$$D_{11} = \frac{\pi^{3/2} n^2 c^2 q v_T}{e_\alpha^2 \langle|\nabla s|\rangle^2 \psi'_{\mathrm{tor}}}\frac{\mathrm{d}s}{\mathrm{d}V}\int_0^\infty \mathrm{d}u\, u^3 \mathrm{e}^{-u^2} \sum_{m_2}\sum_{\mathrm{res}} \left(\tau_b |H_{\mathbf{m}}|^2 \left| m_2 \frac{\partial \omega_b}{\partial \eta} + n\frac{\partial \Omega^3}{\partial \eta}\right|^{-1}\right)_{\eta=\eta_{\mathrm{res}}}, \tag{12}$$

and D_{12} containing an extra factor u^2 in the sub-integrand. Here $\eta_{\mathrm{res}} = \eta_{\mathrm{res}}(u)$ are (generally multiple) roots of Eq. (8) resolved with respect to η.

Numerical implementation, benchmarking results and discussion

Coefficients (12) are computed numerically allowing for the general case of a perturbed tokamak magnetic field in Boozer coordinates. Bounce averages are performed via numerical time integration of zero order guiding center orbits and an effective numerical procedure for root finding for Eq. (8) is realized using the scalings $\omega_b(u,\eta) = u\bar{\omega}_b(\eta)$ and $\Omega_{tB}(u,\eta) = u^2\bar{\Omega}_{tB}(\eta)$ and pre-computation of $\bar{\omega}_b$ and $\bar{\Omega}_{tB}$ on an adaptive η-grid. For testing, a circular concentric flux surface tokamak configuration is used with safety factor shown in Fig. 1. The perturbation field amplitude in Eq. (7) is taken as $B_n(\vartheta) = \varepsilon_M B_0(\vartheta)\exp(im\vartheta)$ with (formally infinitesimal) ε_M set to 1 in all plots. Numerical results for D_{11} are normalized below by the mono-energetic plateau value $D_p = \pi q v_T^3/(16R\bar{\omega}_{c\alpha}^2)$ where R is the major radius, the (0,0) Boozer harmonic is used for the reference frequency $\bar{\omega}_{c\alpha}$ and perturbation wavenumbers (m,n) are indicated in the titles. In Fig. 1 results of Eq. (12) for relatively weak radial electric fields where the superbanana-plateau regime is possible are compared to the analytical large aspect ratio formula of Shaing [1] for a range of aspect ra-

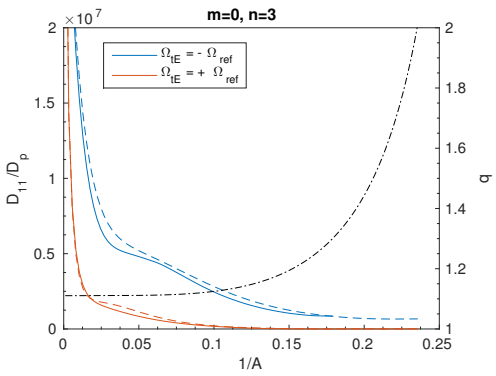

Figure 1: Superbanana plateau - results of Eq. (12) (solid) and analytical formula [1] (dashed). Here $\Omega_{\mathrm{ref}} = cT_\alpha/(e_\alpha\psi'_{\mathrm{tor}})$ and q is the safety factor (dash-dotted).

tio values A. Earlier this formula was shown to be in agreement with computations by the quasilinear version of the drift kinetic equation solver NEO-2 [3] for $A = 10$ and the same

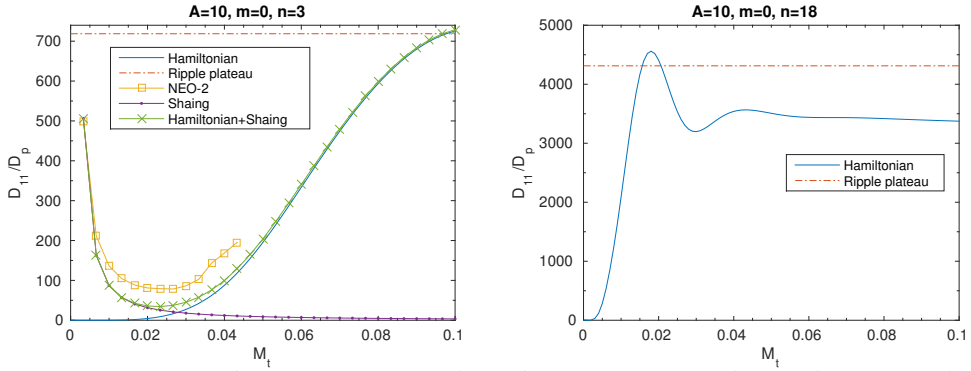

Figure 2: Drift-orbit resonances - Mach number scan for $n = 3$ (left) and $n = 18$ (right).

parameters as in Fig. 1. In Fig. (2) Mach number scans of D_{11} for $A = 10$ are shown for $\Omega_{tB} = 0$ what is valid for relatively "large" Mach numbers $M_t = \Omega_{tE} R/v_T$. Since $m = 0$, passing particles contribute little, which corresponds to the regime of bounce resonances.

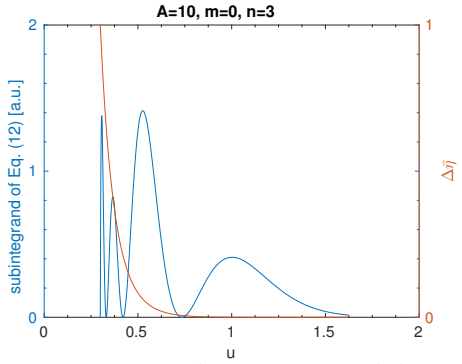

Figure 3: Integrand of Eq. (12) and the resonance line for $M_t = 0.02$ and $m_2 = 1$.

Results are compared to the ripple-plateau regime [7] where $D_{11} = 4\pi^{-1/2} nqA^2 \varepsilon_M^2 D_p$, to the universal formula of Shaing for collisional transport in bounce-averaged regimes [1] and to results of NEO-2. The last two computations were done for very low $\nu^* = 5 \cdot 10^{-5}$ and $n = 3$. The sum of resonant and bounce averaged coefficient is seen to be a factor 2 smaller than the NEO-2 result at intermediate Mach numbers, $M_t \sim 0.02$. The reason can be seen from Fig. 3 where the integrand of Eq. (12) and the normalized resonance condition $\Delta\bar{\eta} = (\eta_{\mathrm{res}}(u) B_{\max} - 1) / (B_{\max}/B_{\min} - 1)$ are plotted for the mainly contributing first bounce harmonic $m_2 = 1$. Significant contributions lie around $u = 1$ where the distance between resonance and trapped-passing boundary is very small, $\Delta\bar{\eta} \sim \exp(-2\pi m_2 u/(qM_t\sqrt{2A}))$.

Conclusion

The proposed unified approach treats all resonant NTV regimes in tokamaks in the same manner. A numerical code has been developed for NTV calculations in these regimes without using model geometry simplifications. A validation for the superbanana plateau regime shows good agreement to Ref. [1]. For drift-orbit resonances, the toroidal torque from NEO-2 [3] is significantly higher than the sum of collisional bounce averaged [1] and resonant regimes. This is due to strong contributions near the trapped-passing boundary. Therefore, collisional boundary layer analysis will be additionally required even at very low ν^*.

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