Numerical Action-Angle transform for Guiding-Center particle motion in Axisymmetric Tokamak equilibria.

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Introduction

The Lagrangian of the Guiding Centre (GC) motion of a charged particle is $\mathscr{L} = \left(\mathbf{A} + \rho_{\|} \cdot \mathbf{B}\right) \cdot \mathbf{v} + \mu\dot{\xi} - H$, where $\mathbf{A}$ and $\mathbf{B}$ are the vector potential and the magnetic field respectively, $\mathbf{v}$ is the guiding centre velocity, μ the magnetic moment, ξ, the gyrophase, $\rho_{\|}$ the parallel velocity to the magnetic field, normalized with B and $H = \rho_{\|}^2 B^2/2 + \mu B + \Phi$ the Hamiltonian, with Φ the electric potential [1]. All quantities are evaluated at the guiding centre and normalized with respect to the nominal magnetic axis gyrofrequency and the major radius R. It has been shown that, when the magnetic coordinate system is appropriately chosen, the dynamical system is Hamiltonian and one can define P_τ and P_χ to be the canonical poloidal and toroidal momenta respectively [2]. In axisymmetric equilibria, the canonical position χ is ignorable and P_χ is conserved, so that the dynamical system, being reduced to one Degree Of Freedom (DOF), is integrable. However, the motion in phase space is non–trivial and there is no straightforward way to predict the behaviour of the system when perturbations are introduced and the integrability is lost.

In this paper we present a method for transforming any dynamical system describing the GC motion of a given axisymmetric equilibrium to an action angle (AA) phase space, where the study of the dynamics is significantly simplified. We demonstrate how this procedure makes it remarkably easy to pinpoint the location and the resonances in the presence of perturbations, as well as determine the conditions for nonlinear interaction due to stochastization, synergy between different perturbations, and confinement loss.

The Action Angle Transform

The conserved canonical momenta P_χ and μ are already the actions of the toroidal motion and the gyromotion respectively. The AA pair (J, θ) of the poloidal motion is found by integrating along a closed orbit in the poloidal plane. By virtue of the Liouville–Mineur–Arnlold theorem, such a transform always exists locally. In particular we can cover all phase space with a measurable set of AA transforms, one for each phase space region that is bounded by a separatrix. From now on we shall call such a region a *continent*, while the set of transforms for all regions is called an *atlas*. In each continent such a transform is implicitly generated by a generating function of the form $F_2 = F\left(\tau, J, P_\chi, \mu\right)$. Dependence of F_2 on P_χ and μ implies that χ is also transformed to an angle variable

$$\bar{\chi} = \chi - f_\chi\left(J, \theta, P_\chi, \mu\right) \tag{1}$$

and so does ξ. Therefore, this procedure generates the transform to the AA pairs $\left(P_\chi, \bar{\chi}\right)$ and $\left(\mu, \bar{\xi}\right)$. In the AA phase space, the actions J, P_χ and μ remain constant, while the angles θ, $\bar{\chi}$ and ξ evolve linearly in time, with frequencies ω_θ, $\omega_{\bar{\chi}}$ and Ω_c respectively.

Orbital spectrum analysis

A perturbation of the form $\delta\mathbf{B} = \nabla \times \sigma\mathbf{B}$ can be straightforwardly included in the guiding centre Hamiltonian as $H = (\rho_c - \sigma)^2 B^2/2 + \mu B + \Phi$, $\rho_c = \rho_\parallel + \sigma$. This introduces a first and second order perturbation Hamiltonian term. Any perturbation with physical meaning should be given in terms of the magnetic coordinates, or even the lab coordinates. The first order Hamiltonian H_1 is proportional to σ. Due to the nonlinear depencence of $\bar{\chi}$ on θ (eq. 1), a monochromatic mode $\sigma = A_{m,n}(\psi)\exp\left(i\left(m\chi + n\tau - \omega t\right)\right)$ in the magnetic coordinates gives an infinite series of modes in the AA phase space, so that

$$H_1 = \sum_s \mathscr{H}^1_{s,m}\left(J, P_\chi\right)\ e^{i(m\bar{\chi} + s\theta - \omega t)}, \tag{2}$$

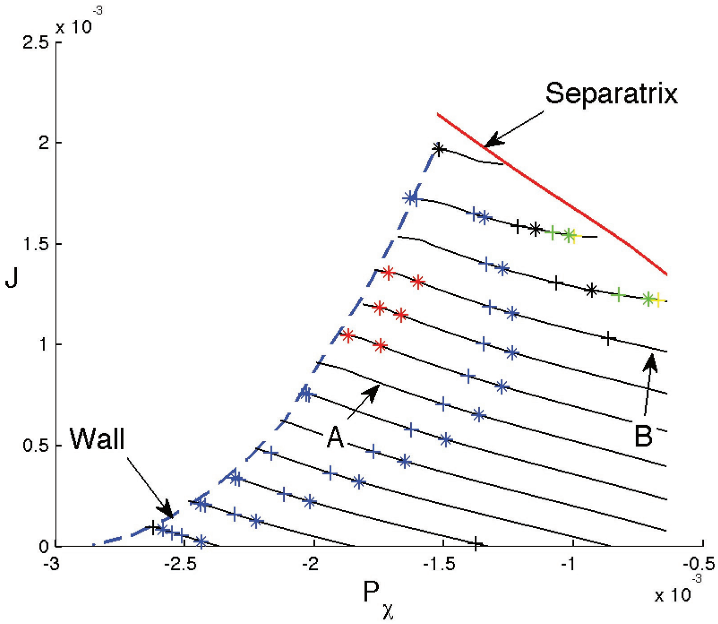
where

$$\mathscr{H}^1_{s,m}\left(J, P_\chi\right) = \frac{1}{2\pi} \oint_{J, P_\chi = \text{const.}} H^1_{m,n}(\psi)\ e^{i\left(n\tau + nf_\chi\left(J, P_\chi, \theta\right) - s\theta\right)} d\theta. \tag{3}$$

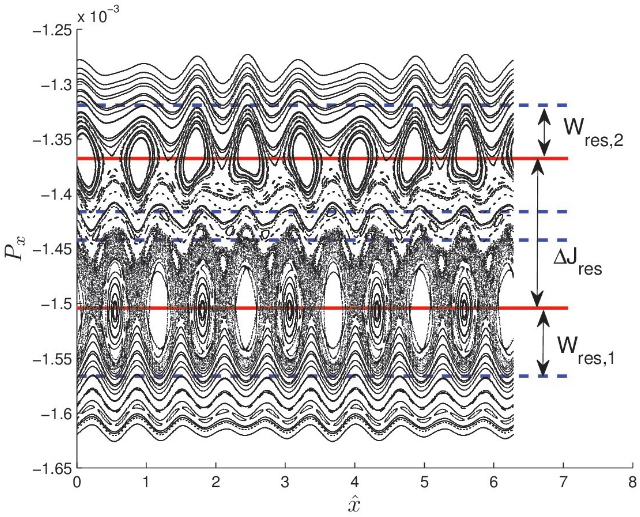
As equation eq. 2 indicates, the resonances of the perturbation are located in action space at the points where the resonance condition

$$m\ \omega_{\bar{\chi}}\left(J, P_\chi, \mu\right) + s\ \omega_\theta\left(J, P_\chi, \mu\right) - \omega\ = 0 \tag{4}$$

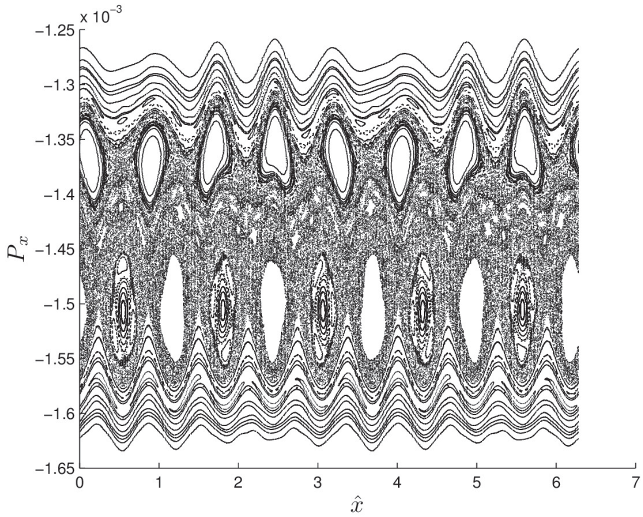
is met. The location of the resonances in action space depends only on the spectral parameters m and ω. The actual profile of the perturbation, i.e the depencence on n or ψ, is relevant in defining the amplitude of the resonant terms, but not in pinpointing their location in the orbital spectrum. Since s can take on any integer value, each bounded continent contains an infinite number of such frequencies, most of which are located in the narrow chaotic sea near the separatrix, where ω_θ approaches zero. In the bulk of each continent there are only a few, if any, sites where eq. 4



(a) Resonance chart. The solid black lines depict the energy surfaces, crosses and stars correspond to resonances with $m = 10$ and $m = 8$ respectively.



(b) Poincare plot on the surface A of Fig 1a for two modes with subcritical amplitude. The semi-analitically calculated positions of the resonances as well as their widths are denoted with solid and dashed lines respectively.



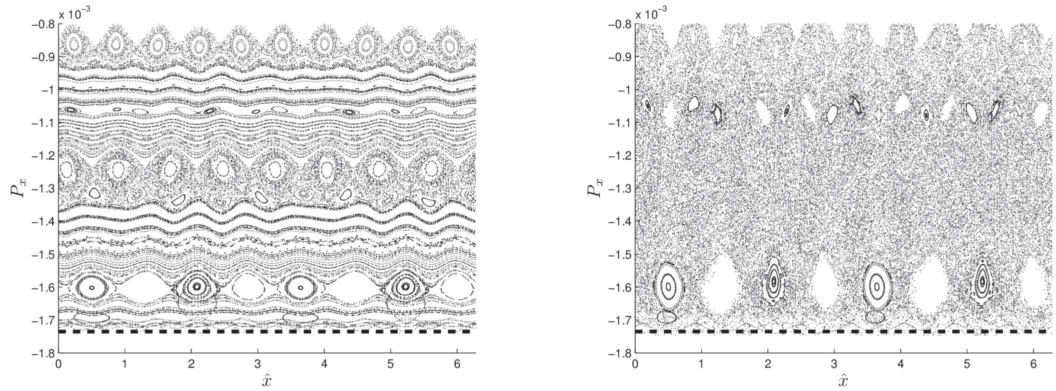
(c) The same Poincare plot for perturbations with critical amplitude. KAM lines between the two resonances have been destroyed and significant redistribution can take place.

Figure 1: Inspection of the resonance chart can reveal the phase space regions when mode synergy can be significant. The analytically calculated resonance positions, width and overlap conditions are in excellent agreement with the simulations.

is satisfied.

The case of time independent perturbations is particularly simple, but rather indicative of the power of the AA transform and the orbital spectrum transform. Near a particular resonance $m\ \omega_{\bar{\chi}} + s\ \omega_\theta = 0$, the dynamics follow a pendulum– like Hamiltonian and a trapped area of width proportional to the square root of the perturbation amplitude is formed. The width depends on $\mathscr{H}_{s,m}\left(J,P_\chi\right)$ and can be easily calculated once the AA transform has been performed. The Hamiltonian is conserved and the quantity $P_\chi - m/sJ$ is an adiabatic invariant. The cases where the ratio m/s or s/m becomes very large are of little interest, since the adiabatic invariant coincides with one of the actions, so that no significant redistribution takes place.

Coexistence of more than one resonances on the same energy surface can lead to chaotic redistribution, due to destruction of the adiabatic invariant. Chiricov criterion, which relies on resonance overlap, provides a very good estimation of the required conditions for chaotic motion. Figure 1a depicts the chart of the $m = 10$ and $m = 8$ resonances in a banana continent of a peaked large aspect ratio equilibrium. As demonstrated by Figs. 1b, 1c, inspection of the resonance chart and employment of the orbital spectrum transform (Eq. 3) provides useful insight on the dynamics of the perturbed phase space.



(a) Poincare cut for the energy surface B of Fig 1a and subcritical amplitude 0.08 a_{Chirikov}. Only two of the resonances have partially overlaped.

(b) The same, with amplitude 0.3 a_{Chirikov}. Although, this is still subctitical, the KAM surfaces have been destroyed. Chirikov criterion overestimates the critical amplitude, by ignoring higher order resonances.

Figure 2: The AA transform as a tool for estimating conditions for confinement loss. The outer closed flux surface is marked with a thick dashed line.

The case of the energy surface B in Fig 1a is of particular importance, because there is a line of resonances linking a deeply trapped part of phase space to the plasma wall (dashed blue line), provided that all consecutive resonances overlap. Chirikov criterion determines the critical amplitude at $a_{crit} = 1.2 \cdot 10^{-3}$,which is an overestimation, due to the strong presence of higher order resonances, as demonstrated in Fig. 2.

References

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