Multiple scattering of radio frequency waves by blobs: Homogenization of a mixture of blobs and the Waterman-Truell approach(*)

K. Hizanidis[1], S. I. Valvis[1], F. Bairaktaris[1] , Gr. Zouros[1], G. Kokkorakis[1], J. Roumeliotis[1]

[1]*National Technical University of Athens, Athens, Greece*

A. K. Ram[2]

[2]*Plasma Science and Fusion Center, MIT, Cambridge MA, USA*

Radio frequency waves are of tantamount importance for heating and current drive in magnetized fusion plasmas. The scattering process of these waves by a multitude of density fluctuations, such as blobs in the edge region, is studied by homogenizing the part of the edge region which is populated by an ensemble of ellipsoidal blobs immersed in the ambient plasma. For that region, the effective permittivity tensor is formulated on the basis of the depolarization dyadic. The ambient density is considered as step-wise constant. In general, the interfaces between the slab characterized by the effective permittivity tensor and the ambient plasma are not necessarily aligned with the ambient magnetic field which is considered as homogeneous.

From now on, the scattering problem of an incident wave is amenable to two different approaches: (a) by considering the scattering and transmission through a composite slab surrounded by the ambient plasma and (b) by a suitably adapted version of the Waterman-Truell approximation formula for the effective wave number in conjunction with the fulfillment of a self consistency condition for the mean field.

(a) Scattering and transmission through a composite slab

From this point of view, the waves propagate in three regions. The first one is homogeneous and anisotropic and consists of ions with number density n_1. Then, the incident wave propagates in an intermediate dielectric mixture plasma region which consists of a homogeneous anisotropic environment in which the ions have number density n_3 and anisotropic spherical inclusions (blobs) with ion number density n_B. After getting through this intermediate region, the wave propagates in a homogeneous anisotropic region consisting of ions with number density n_2. Furthermore, the axis of the intermediate region is not necessarily parallel to the external imposed magnetic field $\vec{\mathbf{B}}_0$ (figure 1).

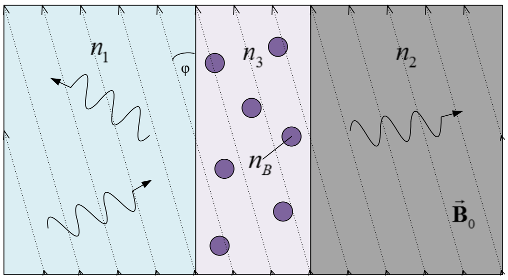

Figure 1: A simplified depiction of the problem – the three propagation regions

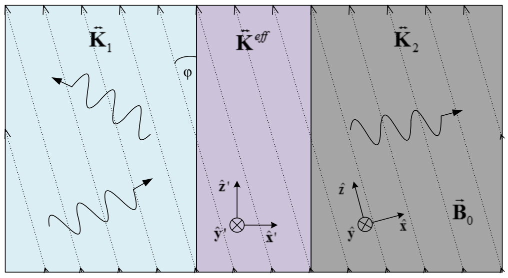

Figure 2: Intermediate region is considered as a dielectric region with calculated effective tensor $\overleftrightarrow{\mathbf{K}}^{eff}$

So, in the intermediate propagation region, there are anisotropic spherical inclusions with an arbitrary permittivity tensor $\overleftrightarrow{\mathbf{K}}^{B}$ located in an homogeneous anisotropic environment with a permittivity tensor $\overleftrightarrow{\mathbf{K}}^{P}$. This problem, can be simplified by replacing the region of plasma with spherical blobs with a dielectric region with effective permittivity tensor $\overleftrightarrow{\mathbf{K}}^{eff}$ (figure 2) [1], [2].

$$\overleftrightarrow{\mathbf{K}}^{eff} = \tilde{\overleftrightarrow{\mathbf{I}}} \cdot \overleftrightarrow{\mathbf{K}}^{P}, \;\; \tilde{\overleftrightarrow{\mathbf{I}}} \equiv \overleftrightarrow{\mathbf{I}} + f\left(\overleftrightarrow{\mathbf{K}}^{B} - \overleftrightarrow{\mathbf{K}}^{P}\right) \cdot \left[\overleftrightarrow{\mathbf{K}}^{P} + \left(1-f\right)\overleftrightarrow{\mathbf{N}} \cdot \left(\overleftrightarrow{\mathbf{K}}^{B} - \overleftrightarrow{\mathbf{K}}^{P}\right)\right]^{-1}$$

where $\overleftrightarrow{\mathbf{K}}^{eff,B,P} = \begin{pmatrix} K_{\perp}^{eff,B,P} & -jK_{\times}^{eff,B,P} & 0 \\ jK_{\times}^{eff,B,P} & K_{\perp}^{eff,B,P} & 0 \\ 0 & 0 & K_{\parallel}^{eff,B,P} \end{pmatrix}$ and $\tilde{\overleftrightarrow{\mathbf{I}}} = \begin{pmatrix} \tilde{I}_{\perp} & -j\tilde{I}_{\times} & 0 \\ j\tilde{I}_{\times} & \tilde{I}_{\perp} & 0 \\ 0 & 0 & \tilde{I}_{\parallel} \end{pmatrix}$.

The depolarization matrix is $\overleftrightarrow{\mathbf{N}} = \begin{pmatrix} N_{\perp} & 0 & 0 \\ 0 & N_{\perp} & 0 \\ 0 & 0 & N_{\parallel} \end{pmatrix}$ with $N_{\perp} = \left(1-N_{\parallel}\right)/2$ and f is the volume fraction that the inclusions occupy, while:

I.) $K_{\parallel}^{P} > K_{\perp}^{P}$: $N_{\parallel} = \dfrac{K_{\parallel}^{P}/K_{\perp}^{P}}{\sqrt{K_{\parallel}^{P}/K_{\perp}^{P}-1}^{3}}\left(\sqrt{K_{\parallel}^{P}/K_{\perp}^{P}-1} - \arctan\sqrt{K_{\parallel}^{P}/K_{\perp}^{P}-1}\right)$

II.) $K_{\parallel}^{P} < K_{\perp}^{P}$: $N_{\parallel} = \dfrac{K_{\parallel}^{P}/K_{\perp}^{P}}{2\sqrt{1-K_{\parallel}^{P}/K_{\perp}^{P}}^{3}}\left(\ln\dfrac{1+\sqrt{1-K_{\parallel}^{P}/K_{\perp}^{P}}}{1-\sqrt{1-K_{\parallel}^{P}/K_{\perp}^{P}}} - 2\sqrt{1-K_{\parallel}^{P}/K_{\perp}^{P}}\right)$

and

$$\tilde{I}_{\perp} = 1 + f\frac{K_{\perp}^{P}\left(K_{\perp}^{B} - K_{\perp}^{P}\right) - K_{\times}^{P}\left(K_{\times}^{B} - K_{\times}^{P}\right) + \left(1-f\right)N_{\perp}\left[\left(K_{\perp}^{B} - K_{\perp}^{P}\right)^{2} - \left(K_{\times}^{B} - K_{\times}^{P}\right)^{2}\right]}{\left[K_{\perp}^{P} + \left(1-f\right)N_{\perp}\left(K_{\perp}^{B} - K_{\perp}^{P}\right)\right]^{2} - \left[K_{\times}^{P} + \left(1-f\right)N_{\perp}\left(K_{\times}^{B} - K_{\times}^{P}\right)\right]^{2}}$$

$$\tilde{I}_{\parallel} = 1 + f\frac{K_{\parallel}^{B} - K_{\parallel}^{P}}{K_{\parallel}^{P} + \left(1-f\right)N_{\parallel}\left(K_{\parallel}^{B} - K_{\parallel}^{P}\right)}, \quad \tilde{I}_{\times} = f\frac{K_{\perp}^{P}K_{\times}^{B} - K_{\times}^{P}K_{\perp}^{B}}{\left[K_{\perp}^{P} + \left(1-f\right)N_{\perp}\left(K_{\perp}^{B} - K_{\perp}^{P}\right)\right]^{2} - \left[K_{\times}^{P} + \left(1-f\right)N_{\perp}\left(K_{\times}^{B} - K_{\times}^{P}\right)\right]^{2}}$$

Also, in a more realistic approach, the initial spheroidal shape of the scattering regions is distorted into ellipsoidal during an affine transformation necessary to keep the potential outside distorted into ellipsoidal during an affine transformation necessary to keep the potential outside the sphere obeying the Laplace equation [2]. In this case, the depolarization matrix components are given by the integral $N_x = \frac{a_x a_y a_z}{2}\int_0^{\infty}\left(s + a_x^{\,2}\right)^{-1}\left[\left(s + a_x^{\,2}\right)\left(s + a_y^{\,2}\right)\left(s + a_z^{\,2}\right)\right]^{-\frac{1}{2}} ds$, where a_x, a_y, a_z are the semiaxes of the ellipsoidal inclusion. This hyperelliptic integral can be expressed as function of known elliptic integrals [3].

In a coordinate system (*x*, *y*, *z*) with the *z*-axis aligned with the ambient magnetic field, one needs to rotate around *y* to the one aligned with the surfaces of the slab (*x'*, *y'*, *z'*) using the rotation matrix $\overleftrightarrow{\mathbf{R}}$, via $\left(\overleftrightarrow{\mathbf{K}}^{B,P,eff}\right)' = \overleftrightarrow{\mathbf{R}} \cdot \overleftrightarrow{\mathbf{K}}^{B,P,eff} \cdot \overleftrightarrow{\mathbf{R}}^{T}$. Furthermore, the propagation vector of the incident wave must be expressed in the new aligned coordinate system. Choosing the plane of incidence to be the (x-z) plane [the containing the blobs slab is infinite in the (y-z) directions] and following Berreman [4], one may write the two vectorial Maxwell's equations for the phasors of the electromagnetic field as $\left(\overleftrightarrow{\mathbf{M}} + j\frac{\omega}{c}\overleftrightarrow{\mathbf{C}}\right)\cdot\vec{\mathbf{F}} = 0$, where $\overleftrightarrow{\mathbf{M}}$ consists of spatial derivatives with respect to *x, z*, $\overleftrightarrow{\mathbf{C}}$ contains the permittivity tensor components and the rotation angle *φ* and $\vec{\mathbf{F}}$ includes the six components of the electric and the magnetic field. Then, since the properties of the various regions involved change along the *x*-direction, one may Fourier transform the differential equation in the *z*-direction accordingly [exp($jk_{||}z$)] and therefore $\left(\overset{\frown}{\overleftrightarrow{\mathbf{M}}} + j\overleftrightarrow{\mathbf{C}}\right)\cdot\overset{\frown}{\vec{\mathbf{F}}} = 0$ where the arc on top signifies Fourier transform in the *z*-direction so that the differentiation with respect to *z* changes to a simple algebraic operation related to the *z* component of the wavevector. The latter linear problem is solved in the three domains of Fig. 2 where the density can be taken as piece-wise constant.

(b) Waterman-Truell approach

This approach is limited to spherical blobs. However, it is capable of incorporating multiple scattering effects. In its algorithmic implementation involves an iterative procedure where the previous method for the calculation of the effective dielectric tensor of the effective medium is required. It also involves the solution of two scattering problems, namely the propagation of the incident wave in the effective medium and scattered by (1) a spherical blob of density which is equal to the equivalent to the composite slab (solved previously) and (2) by a spherical blob of blob density. After solving these two scattering problems using the method of Ram and Hizanidis [5], a test takes place to find out if a consistency condition referring to the mean fields and the wavenumbers is satisfied. If it is, the procedure is considered completed and it ends. If it's not, then the wave vector is being replaced with a more exact using the Waterman and Truell method [6], the two scattering problems are being solved again and the procedure repeats itself until the consistency condition is satisfied.

(*)Supported in part by the Hellenic National Programme on Controlled Thermonuclear Fusion associated with the EUROfusion Consortium

[1] A. K. Ram et al., Phys. Plasmas 20, 056110 (2013).

[2] Ari Sihvola, Homogenization of a dielectric mixture with anisotropic spheres in anisotropic background (1996), Department of Electroscience, Electromagnetic Theory, Lund Institute of Technology, Sweden

[3] Gradshteyn / Ryzhik, Table of Integrals, Series, and Products (1980), Academic Press Inc., New York, USA

[4] D. W. Berreman (1972), Optics in Stratified and Anisotropic Media: 4x4-Matrix Formulation, Journal of the Optical Society of America, 62:4, 502-510, DOI:10.1364/JOSA.62.000502

[5] Abhay K. Ram & Kyriakos Hizanidis (2013) Scattering of electromagnetic waves by a plasma sphere embedded in a magnetized plasma, Radiation Effects and Defects in Solids: Incorporating Plasma Science and Plasma Technology, 168:10, 759-775, DOI: 10.1080/10420150.2013.835633

[6] A. Lakhtakia (1993) Application of the Waterman-Truell approach for chiral composites, International Journal of Electronics, 75:6, 1243-1249, DOI: 10.1080/00207219308907198