Multiresolution and fast multipole methods for streamer simulations

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Introduction

This contribution presents a new numerical strategy for the computation of the electric field in multi-scale streamer discharge simulations. The electric field is determined by using an integral equation method to solve the Poisson's equation and accelerated by a fast multipole method (FMM) [1, 2, 3, 4]. An adaptive mesh refinement technique based on space adaptive multiresolution (MR) [5, 6] is implemented to obtain a hierarchy of embedded adapted grids (a binary tree in the case of one spatial dimension). To show the interest of the method, we present the simulation of a positive streamer discharge by means of a 1.5D model, where the spatio-temporal evolution of the charged particle densities is solved only along one spatial dimension and the electric field is determined in three dimensions using a disc method [7]. Even though a similar numerical strategy can be straightforwardly applied to higher spatial dimensions [8], the numerical efficiency of the electric field computation in a 1.5D model can be advantageous, as this model can be considered, for instance, to investigate thermal electron acceleration [9] or different kinetic schemes in streamer discharges [10]. All the details on the positive streamer model considered here as well as the numerical strategy to solve the spatio-temporal evolution of the particle densities can be found in [7], while for the sake of brevity here we focus only on the computation of the electric field.

Computation of the electric field

The 1D computational domain D is divided into a set of cells of different spatial resolution. The source distribution $\rho_i(y)$ is thus given on N leaf nodes D_i. Hence, $D = \cup_{i=1}^{N} D_i$ and the electric field from a charged cylinder of radius R at the target

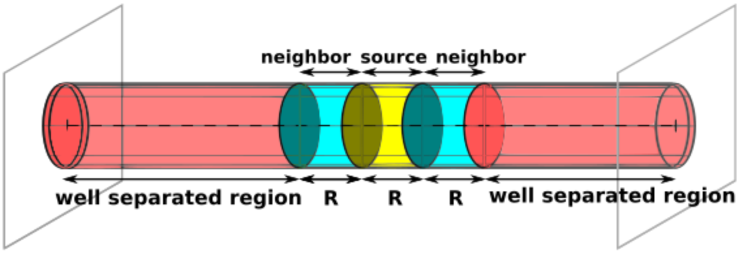

Figure 1: The multipole expansion is convergent in the well separated region from the source panel. R is the radius of the cylinder.

point x on the discharge axis is given by

$$E(x) = \frac{1}{2\varepsilon_0}\sum_{i=1}^{N}\rho_i\int_{D_i}K(x,y)dy, \tag{1}$$

where

$$K(x,y) = \operatorname{sign}(x-y) - \frac{x-y}{\sqrt{(x-y)^2+R^2}}. \tag{2}$$

A direct evaluation of (1) over a set of N target points x_j clearly requires $O(N^2)$ computations. The FMM relies on an approximation to the kernel $K(x,y)$ by a p-term expansion, with coefficients $\psi_k(R,y)$ depending on source points alone, and convergent in the well separated region from the source cylinder (see figure 1), i.e., for $|x| > \sqrt{y^2+R^2}$ [3, 4]. Then, we have

$$K(x,y) \approx \operatorname{sign}(x-y) - \operatorname{sign}(x)\left(1-\sum_{k=1}^{p}\psi_k(R,y)x^{-k}\right) = \operatorname{sign}(x-y) -$$
$$-\operatorname{sign}(x)\left(1-\frac{1}{2}R^2x^{-2}-yR^2x^{-3}-\frac{1}{8}(12y^2R^2-3R^4)x^{-4}-\dots\right). \tag{3}$$

The approximation to the electric field at x can be then expressed as a superposition of local and far field contributions:

$$E(x) \approx E_{\mathrm{loc}}(x)+\frac{\operatorname{sign}(x)}{2\varepsilon_0}\sum_{k=1}^{p}\sum_{i=1}^{N_{\mathrm{ff}}}\rho_i\left(\int_{D_i}\psi_k(R,y)dy\right)x^{-k} = E_{\mathrm{loc}}(x)+\frac{\operatorname{sign}(x)}{2\varepsilon_0}\sum_{k=1}^{p}\alpha_k x^{-k}, \tag{4}$$

where the moments α_k depend only on the distribution of N_{ff} sources in well separated cells and can be precomputed, while $E_{\mathrm{loc}}(x)$ requires direct integration. Moreover, well

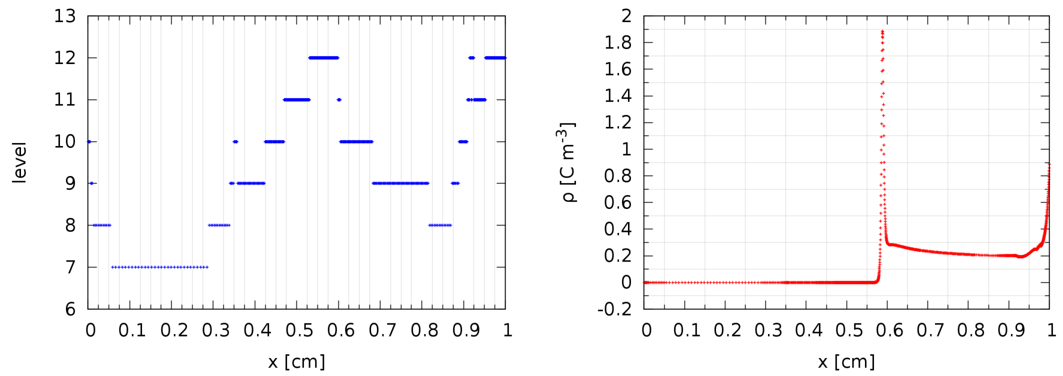

Figure 2: Adapted grid (left) corresponding to a given charge density (right) for an MR tolerance of $\eta = 10^{-6}$.

separated regions are taken into account on the coarsest level possible regarding the hierarchy of grid refinements. This allows in particular to take into account efficiently image charges required for Dirichlet boundary conditions. Consequently, the evaluation of E at each leaf node involves $O(pN)$ operations.

Results

Positive streamers are simulated as in [7]. The electric field is computed by means of the FMM on an MR adapted grid. Grid levels corresponding to a given charge distribution at 6 ns are shown in figure 2, for an MR tolerance of $\eta = 10^{-6}$.

We compare the CPU time for the FMM, t_{fmm}, with the one required in the direct computation, t_{dir} (evaluation of eq. (32) in [7]). Note that the FMM also involves direct computations, but only for the local contribution. Table 1 shows t_{dir} and t_{fmm} for several MR tolerances η and $p = 25$, and thus adapted grids of different size $\#AG$ ($\#AG = 4096$ corresponds to a uniform grid at level 12). The speedup obtained with the FMM depends on the number of cells in the adapted grid. Further analysis shows that the com-

Table 1: Computational results (CPU times t_{dir} and t_{fmm} in ms).

η	#AG	t_{dir}	t_{fmm}	speedup
10^{-3}	168	6	2.6	2.3
10^{-4}	349	26	6.3	4.1
10^{-5}	635	87	13	6.6
10^{-6}	1064	244	26	9.4
10^{-7}	2098	947	66	14.3
10^{-8}	2809	1696	97	17.5
10^{-9}	3232	2260	110	20.5
10^{-10}	3900	3340	140	23.8
—	4096	3600	150	24.0

putational complexity of the direct method is indeed of $O(N^2)$, whereas the FMM behaves like $O(N^{1.3})$ with $p = 25$. This departure from linear complexity may be due to the relatively high number of local computations needed to ensure convergence depending on the streamer radius R. Nevertheless, computational speedup is significant even for cases with very low number of cells. This is achieved also thanks to the higher efficiency of the FMM when handling boundary conditions requiring image charges.

Conclusion

A new approach was introduced to compute electric field in streamer simulations, where fast multipole acceleration was implemented on an adapted grid resulting from space adaptive multiresolution. Moreover, the FMM allows one to account for image charges needed for Dirichlet boundary conditions very efficiently. The same numerical strategy can be straightforwardly extended to higher spatial dimensions, where an even better computational efficiency is expected.

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